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Non-central *t* distribution and the power of the *t* test : A rejoinder

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Non-central *t* distribution needed for assessing the power of the *t* test is described. Three approximations are compared and their merits discussed in regard to simplicity and accuracy.

Power evaluation for the various forms of Student's *t* test ties in with the non-central *t* distribution, explicitly noted $t'_\nu(\delta)$, where ν is the "degrees of freedom" parameter and δ the non-centrality value. The $t'_\nu(\delta)$ variable represents the quotient of a standard normal (z) variable displaced by a constant (δ), over the square root of a Chi-square (χ^2) variable divided by its parameter (ν) :

$$t'_\nu(\delta) = \frac{z+\delta}{\sqrt{\chi^2_\nu / \nu}}\,. \qquad (1)$$

$t'_\nu(0)$ coincides with the standard (central) t_ν distribution. In the following, δ represents the effect size; by convention, a δ of 0.5 is considered a "medium" effect size.

In Figure 1 are shown three instances of the t' density envelope for $\nu = 10$, one with $\delta = 0$ (a standard t), and two with $\delta = 3$ and $\delta = 6$: one may note that, whereas $t_{10}(\delta = 0)$ is centered at 0 and symmetrical, the non-central t''s are displaced toward δ, their variance is increased and they are skewed.

The probability density function for $t'_\nu(\delta)$ is (Levy & Narula, 1974) :

$$f(t) = c\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\nu+k+1}{2}\right)\delta^k 2^{k/2} t^k}{\Gamma(k+1)(\nu+t^2)^{k/2}} \qquad (2)$$

in which

$$c = \frac{\nu^{\nu/2}}{\sqrt{\pi}\,\Gamma(\nu/2)} \times \frac{e^{-\delta^2/2}}{(\nu+t^2)^{(\nu+1)/2}}. \qquad (2)$$

Its first statistical moments are:

$$\mu'_1 = \delta\sqrt{\frac{\nu}{2}}\,\Gamma\left(\frac{\nu-1}{2}\right)\left[\Gamma\frac{\nu}{2}\right]^{-1}, \qquad (3a)$$

$$\mu_2 \equiv \sigma^2 = \nu/(\nu-2)(1+\delta^2) - (\mu'_1)^2, \qquad (3b)$$

$$\mu_3 = \mu'_1\left[\frac{\nu(2\nu-3+\delta^2}{(\nu-2)(\nu-3)}\right] - 2\mu_2, \qquad (3c)$$

$$\mu_4 = \frac{\nu^2(3+6\delta^2+\delta^4)}{(\nu-2)(\nu-4)} - {\mu'_1}^2\left[\frac{\nu(\nu+1)\delta^2+3(3\nu-5)}{(\nu-2)(\nu-3)} - 3\mu_2\right] \qquad (3d)$$

in which μ denotes the moments about the mean (in particular, μ_2 is the variance) and μ' denotes the central moments (in particular, μ_1 is the mean, also noted E(t')).

Note that, approximately, $\mu_1' \approx \delta \times [1 + 14/(17\nu)]$ and $\sigma^2 \approx \nu/(\nu-2) + \delta^2/(2\nu-7)$. It can be shown for given δ that, as ν increases, $E(t') \to \delta$, $\mathrm{var}(t') \to \mathrm{var}(t) = \nu / (\nu - 2)$, skewness index $\gamma_1(t') = \mu_3 / \sigma^3 \to \gamma_1(t) = 0$ and kurtosis index $\gamma_2(t') = \mu_4 / \sigma^4 - 3 \to \gamma_2(t) = 6 / (\nu - 4)$. Of course, we also know that $t \to z$ with increasing ν, z being a standard normal variable. These relations have inspired some approximation procedures. For illustration, Table 1 shows calculated values of the moments μ_1', σ^2, γ_1 and γ_2 for some combinations of ν and δ, together with the above approximations for μ_1' and σ^2.

Power calculation and approximations

A standard reference for statistical distributions is the celebrated series by Johnson, Kotz and Balakrisnan (1994, 1995), of which Volume 2 devotes a chapter to the non-central *t* distribution. From expression 31.11′ on p. 514, one obtains the following function for evaluating the distribution function of $t'_\nu(\delta)$:

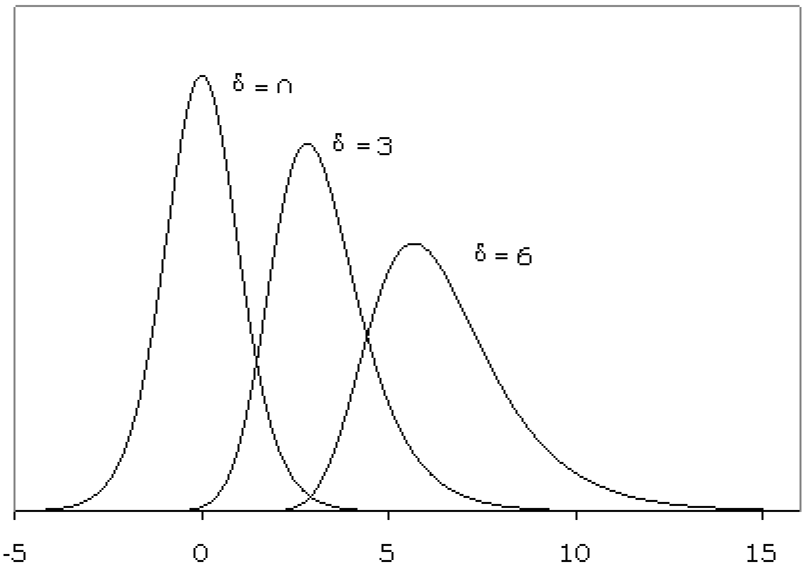

Figure 1. Three non-central t distribution (t') with ν = 10 degrees of freedom and non-centrality values δ = 0, 3 and 6.

$\Pr\{t'_\nu \leq t\}$ =

$$\frac{1}{2^{\nu/2-1}\Gamma(\nu/2)} \times \int_0^\infty x^{\nu-1}e^{-x^2/2}\Phi(tx/\sqrt{\nu}-\delta)dx, \quad (4)$$

where $\Phi(x)$ is the standard normal distribution function. Use of

$$\Pr\{t'_\nu \geq t(\nu[\alpha])\}, \quad (5)$$

where $t(\nu[\alpha])$ is the appropriate critical value for the t test, produces the exact power value.[1]

Cousineau (2007) proposed an approximation to $t'(\delta)$ as $t + \delta$, the power being estimated through:

$$\Pr\{ t_\nu \geq t(\nu[\alpha]) - \delta \} ; \quad (6)$$

he illustrated this with an example involving the comparison of two groups, each with $N_1 = N_2 = 64$ elements, and therefore, a t test having $v = 126$ degrees of freedom. Farther on in the same direction, one could approximate $t'_\nu(\delta)$as $z + \delta$ and obtain an approximate power through :

$$\Pr\{ z \geq z[\alpha] - \delta \} , \quad (7)$$

$z[\alpha]$ being the appropriate critical value from the normal distribution. Finally, Johnson et al. (1995, eq. 31.25) report an approximation by Jennet and Welch (1939), from which we propose the following : [2]

$$\Pr\{ z \geq z^* \} , \quad (8)$$

where

$$z^* = \left[t_{\nu[\alpha]}/\sqrt{\nu}\cdot E(\chi) - \delta\right] / \sqrt{1 + t^2_{\nu[\alpha]}/\nu \cdot var(\chi)}, \quad (8a)$$

$$E(\chi) \approx \sqrt{\nu}\,[1 - 1/(4\nu)], \quad (8b)$$

and $$var(\chi) \approx (4\nu - 1)/(8\nu). \quad (8c)$$

Table 2 presents some illustrative data for comparing these approximations.

Comparing the approximations (6), (7) and (8) to the exact power value (5) highlights the obvious superiority of Jennet and Welch's formula (5), a formula which has the additional advantage of transferring a t' evaluation problem to the well-known and much tabulated standard normal integral. Cousineau's approximation (6), which implies the evaluation of a standard t distribution function, is the next best, more so when degrees of freedom are higher and non-centrality parameter is low. Lastly, except for situations involving very numerous degrees of freedom, the simple normal approximation (7) is not worth considering.

Example 1

As our first example, we examine a situation where a sample of 10 male executives, aged 30-40 years, underwent a thorough physical examination, including VO_2 max evaluation, for which they averaged 43.0 ml.min^{-1}.kg^{-1} O_2 with a standard deviation of 3.5. The mean value of North-American males in that age span is 45.0 (fictitious). The one-sample t test for this situation is $t = (43.0 - 45.0) / 3.5 \approx -1.807$. The two-tailed critical values on a standard t distribution having $\nu = N - 1 = 9$ df and 5% significance level are ± 2.262, so that quite obviously the observed difference is not statistically significant.

Some documentation (fictitious) reports a singular relative effect size δ_1 = -0.2 for this comparison, i.e. each individual in that age group and with similar characteristics (educated, sedentary worker) would deviate by -0.2

[1] Evaluation for the right-hand side, i.e. $t'_\nu \geq t(\nu[\alpha])$, is illustrative of a positive "true effect size" δ > 0. Power for a negative effect δ < 0 would be obtained through $t'_\nu \leq t(\nu[\alpha])$.

[2] Formulas for E(χ) and var(χ) of the χ_ν (square root of χ^2) distribution are approximate.

Table 1. Moments of $t'_\nu(\delta)$ for some combinations of ν and δ, and their approximations

ν	δ	μ_1'	σ^2	γ_1	γ_2	$\approx\mu_1'$	$\approx\sigma^2$
10	0	0,000	1,250	0	1,000	0,000	1,250
10	2	2,167	1,552	0,724	1,827	2,165	1,558
10	4	4,335	2,459	1,097	2,945	4,329	2,481
10	6	6,502	3,970	1,254	3,580	6,494	4,019
5	5	5,947	7,966	2,840	28,889	5,824	10,000
10	5	5,419	3,139	1,192	3,315	5,412	3,173
20	5	5,198	1,872	0,627	0,950	5,206	1,869
50	5	5,077	1,312	0,273	0234	5,082	1,310

standard deviation from the general mean for that age group. The relevant non-centrality parameter in this case would be:

$$\delta_N = \sqrt{N} \cdot \delta_1, \qquad (9)$$

here, $\delta_{10} = \sqrt{10} \times -0.2 \approx -0.632$. Hence, for evaluating the actual power of our test (with N = 10 participants), we refer to the $t_9'(-0.632)$ distribution and calculate Pr{ $t_9'(-.632) \leq -2.262$ } = 0.0819, calculations being on the left-hand side wherein the null hypothesis is to be contradicted.

From approximation (8), we first obtain $E(\chi) \approx \sqrt{9} \times [1 - 1/(4\times9)] \approx 2.9167$, $\mathrm{var}(\chi) \approx (4\times9 - 1) / (8\times9) \approx 0.4861$, then $x = \{-0.632 - (-2{,}262) / \sqrt{9} \times E(\chi)\} / \{1 + (-2.262^2) / 9 \times \mathrm{var}(\chi)\}^{½} \approx 1.387$ and, finally, $1 - \Phi(1.387) \approx 0.0827$. Cousineau's t solution (6) is simply Pr{ $t_9 \leq -2.262 - (-0.632)$ } = Pr{ $t_9 \leq -1.630$ } ≈ 0.0688, a value obtained by resorting to the standard Student t distribution function. Lastly, the normal approximation (7) is simply Pr{ $z \leq -1.960 - (-0.632)$ } = $\Phi(-1.328) \approx 0.0921$. Although it is not particularly accurate, Jennet and Welch's approximation (5) still keeps its promises.

Example 2

Going back to Cousineau's (2007) illustrative example, we have two groups, each containing 64 participants, the difference of their means to be tested with the independent-groups t test procedure. With $\nu = 64 + 64 - 2 = 126$ *df*, the 5% critical values applicable are ±1.979. The "effet size" proposed for this case, established[3] as $\delta_1 = (\mu_1 - \mu_2) / \sigma\sqrt{2}$, is said to be $0.5 / \sqrt{2} \approx 0.35355$; thus the relevant value of the non-centrality parameter δ_N is $\sqrt{n} \times \delta_1 = \sqrt{64} \times 0.35355 \approx 2.8284$: the true power here is 0.8014. By approximation (8), we have $x \approx -0.847$ and Pr{ $z \geq -0.847$ } = $\Phi(0{,}847) \approx 0.8015$. Cousineau's method, with Pr{ $t_{126} \geq 2.8284 - 1.979$ }, produces 0.8014, and the simpler normal approximation gives 0.8074. The large value of the *df* (or ν) parameter in this case insures a converging agreement on the true power, even with the very simple normal approximation (7).

Table 2. Comparison of power calculations from the exact non-central t distribution and three approximations, in different contexts*

	Exact (5)	*Cousineau (6)*	*Normal (7)*	*Jennet & Welch (8)*
	$N = 10$, $\nu = 9$, $t_{9[0.95]} = 1.833$			
δ = .5	0.119	0.108	0.126	0.119
δ = 1	0.236	0.213	0.260	0.236
δ = 2	0.580	0.565	0.639	0.579
δ = 3	0.868	0.863	0.912	0.869
	$N = 50$, $\nu = 9$, $t_{49[0.95]} = 1.677$			
δ = .5	0.125	0.122	0.126	0.125
δ = 1	0.255	0.251	0.260	0.255
δ = 2	0.628	0.626	0.639	0.628
δ = 3	0.905	0.904	0.912	0.905

* Calculations refer to the one-sample t test on the mean, with a right-tailed significance criterion at the 0.05 level.

[3] Some authors use instead $(\mu_1 - \mu_2) / \sigma$, similarly to the one-sample t where we use $(\mu_1 - \mu) / \sigma$.

Discussion and conclusion

The recent availability of on-line calculators for power evaluation or, in some instances, the direct use of a computer-programmed algorithm for the non-central t's distribution function may supersede the need for approximation formulas (see Appendix). It is our experience however that a simple, handy procedure is always welcomed for such a task, be it to cross-check a calculation or to incorporate it in a software package. In this light, Jennet and Welch's method (8, 8a), supplemented with our estimation functions (8b, 8c), simply do the job, with an accuracy that holds in all situations, provided that df (= ν) ≥ 8. As for Cousineau's (2007) approximation (6), its domain of validity depends on a combination of the δ and ν parameters, and it requires the evaluation of Student t's distribution function.

References

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Appendix follows

Appendix: The non-central *t* distribution in existing software

SPSS gives access to the non-central version of the t distribution with the function ncdf.t (t; ν, δ_N) which computes $\Pr\{t'_\nu \leq t\}$. Hence, with the following instructions ran in a syntax window:

```
compute criticalvalue = idf.t(0.975, 9).
compute power= 1-ncdf.t(criticalvalue, 9, 0.632).
execute.
```

The critical value for a t test with 9 degrees of freedom will be computed, and then, the power at that given critical value will be returned. The result above is 0.082, as in Example 1.

Mathematica does not provide the non-central t distribution as a built-in function but this function can be programmed using these three lines of code:

```
Γ[x_] := Gamma[x]
Φ[x_] := CDF[NormalDistribution[0, 1], x]
```

$$\text{ncdf}[t\_,\ \nu\_,\ \delta\_] := \frac{1}{2^{\nu/2-1}\,\Gamma\left[\frac{\nu}{2}\right]} \times \int_0^\infty x^{\nu-1}\, e^{-x^2/2}\, \Phi\left[t\,x \big/ \sqrt{\nu} - \delta\right] dx \ \text{// N}$$

Afterwards, this function can be used as previously, e.g.

```
power = 1 - ncdf[2.262, 9, 0.632]
```

will return 0.0819213 as in Example 1.