



A Tutorial on Testing the Equality of Standardized Regression Coefficients in Structural Equation Models using Wald Tests with lavaan

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Abstract ■ Comparing the effects of two or more explanatory variables on a dependent variable in structural equation models, with either manifest or latent variables, may be hampered by the arbitrary metrics which are common in social sciences and psychology. A possible way to compare the effects is the comparison of standardized regression coefficients by means of the Wald test. In this tutorial, we show how the Wald test can be used to compare standardized regression coefficients. Moreover, we demonstrate how this can be implemented in R using the lavaan package. Additionally, we provide a convenience function that allows doing a Wald test by only setting up equality constraints. The Wald test, its application and the implementation in R are illustrated by means of worked-examples. We also discuss theoretical aspects and implications when hypotheses about the equality of standardized regression parameters in structural equation models are tested.

Keywords ■ comparing standardized regression coefficients, Wald test, structural equation modeling. **Tools** ■ lavaan.

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Introduction

Usually, regression coefficients in structural equation models (SEMs) are compared in multiple group settings. In these cases, it is of interest whether the effects of two or more explanatory variables on a dependent variable are the same or different in two or more populations. Besides such a setting, researchers may want to compare regression coefficients in a single group setting. In these cases, the researcher wants to compare the effects of two or more explanatory variables on a dependent variable. Drawing on the LISREL all-y notation (see Newsom, 2015, pp. 383-386) and the Mathematical Appendix for a description), the effect of an explanatory variable η_i on a dependent variable η_k , $1 \leq i < k \leq p$, where p is the number of variables, is represented by the regression coefficient β_{ki} . To compare these effects, the researcher is interested in testing the following hypothesis:

$$H_0 : \beta_{k1} = \beta_{k2} = \dots = \beta_{ki}. \quad (1)$$

In this tutorial, we always consider a null hypotheses of this form and related two-sided tests. This hypothesis aims at comparing the effects of the explanatory variables on the dependent variable. To explore the meaning of this hypothesis, we take the interpretation of the unstandardized regression coefficient β_{ki} into account. This interpretation is: If the explanatory variable changes one unit, then the dependent variable changes β_{ki} units. A possibility to test this hypothesis is by means of a likelihood ratio test (LR test). To conduct a LR test, an unrestricted model is estimated at first. Afterwards, equality constraints are imposed on the regression parameters of interest and this second model is also estimated. For each model, its χ^2 -test statistic is determined and a LR test is calculated in form of a χ^2 -difference test. Besides the LR test, there is also the Wald test which can be used to test a hypothesis of this kind. To conduct the Wald Test, an unrestricted model is estimated and afterwards it is tested if the restrictions corresponding to the hypothesis are valid by means of a χ^2 -distributed test statistic. According to Greene (2012,

¹A further test is the score test, in which first a restricted model is estimated and afterwards it is investigated how much the model fit changes when



p. 484), the LR and Wald test are asymptotically equivalent under the null hypothesis.¹

However, the comparison of regression coefficients in SEMs is hampered when the explanatory variables have different metrics. Moreover, it is common in social sciences that most measurements have arbitrary metrics. By metric, we “refer to the numbers that the observed measures take on when describing individuals’ standings on the construct of interest” (Blanton & Jaccard, 2006, p. 27). A consequence of these arbitrary metrics is that a variable can be rescaled. For example, any random variable X with an arbitrary metric can be rescaled by means of the linear transformation $Y = a \cdot X + B$, where a and b are real numbers. The variance of Y is then given by $\text{var}(a \cdot X + b) = a^2 \cdot \text{var}(X)$. Accordingly, for the covariance between X and any other random variable Z , after the linear transformation the covariance becomes $\text{cov}(Y, Z) = \text{cov}(a \cdot X + b, Z) = a \cdot \text{cov}(X, Z)$. If such a transformation is applied to an explanatory variable in a SEM, the consequence is that the size of the regression coefficients changes. This is because in a regression setting, the dependent variable itself is a linear transformation of the explanatory variables. Rescaling an explanatory variable alters its variance which in turn is compensated by altering the regression coefficients such that the variance of the dependent variable remains equal. Analogous considerations can be made if the dependent variable is rescaled. Thus, under “the proposition that if the units of measurement ... are arbitrary then there can be no scientific interest in any aspect of the model which depends on the choice of units” (Bartholomew, Knott, & Moustaki, 2011, p. 65).

As a possible way to circumvent the problem of arbitrary metrics is to use standardized regression coefficients (e.g., Kwan & Chan, 2011). Standardized regression coefficients are calculated by multiplying the unstandardized regression coefficients with the standard deviation of the explanatory variable and by dividing this product with the standard deviation of the dependent variable, or in short

$$\beta_{ki}^S = \frac{\sqrt{\text{var}(\eta_i)}}{\sqrt{\text{var}(\eta_k)}} \cdot \beta_{ki}. \quad (2)$$

Standardizing makes the regression coefficients comparable by “norming” the regression coefficients with the variables’ empirical standard deviations. This also alters the interpretation of the regression coefficients which is for standardized regression coefficients as follows: If the explanatory variable changes one standard deviation, then the dependent variable changes β_{ki}^S standard deviations. It is easy to see how standardization compensates in the case of different metrics. The denominator refers to the standard deviation of the dependent variable, so constant

for all regression coefficients referring to the same dependent variable. Multiplying the numerator with the standard deviation thus levels out effects of different metrics. But, as mentioned by Kwan and Chan (2011), it follows that when the equality of standardized regression coefficients is tested, then in contrast to hypothesis (1) the actual hypothesis is

$$\begin{aligned} H_0 : \frac{\sqrt{\text{var}(\eta_1)}}{\sqrt{\text{var}(\eta_k)}} \beta_{k1} &= \frac{\sqrt{\text{var}(\eta_2)}}{\sqrt{\text{var}(\eta_k)}} \beta_{k2} \\ &= \dots = \frac{\sqrt{\text{var}(\eta_i)}}{\sqrt{\text{var}(\eta_k)}} \beta_{ki}. \end{aligned} \quad (3)$$

Using Equation (2), the hypothesis has the more convenient expression

$$H_0 : \beta_{k1}^S = \beta_{k2}^S = \dots = \beta_{ki}^S. \quad (4)$$

Thus, hypotheses (1) and (4) have a fundamentally different interpretation and they are only equivalent if

$$\frac{\sqrt{\text{var}(\eta_1)}}{\sqrt{\text{var}(\eta_k)}} = \dots = \frac{\sqrt{\text{var}(\eta_i)}}{\sqrt{\text{var}(\eta_k)}}. \quad (5)$$

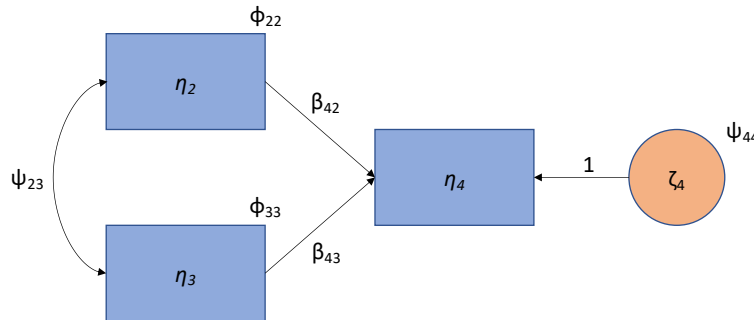
Kwan and Chan (2011) exemplify this issue using a data set of $N = 200$ participants randomly selected from the PISA 2006 study (OECD, 2009). For their example, these authors use three out of five variables, a description of the selected and the other variables and the covariance matrix can be found in Appendix B.1. We use the same numbering of the variables as in Kwan and Chan so that the regression coefficients will have the same indices as in their paper. In their example, Kwan and Chan regress educational resources (η_4) at home on parental education level (η_2) and home possessions (η_3). This model is depicted in Figure 1.

To exemplify an arbitrary metric, we rescaled the variable η_3 by multiplying it by an arbitrarily chosen factor 10, i.e., $\tilde{\eta}_3 = 10 \cdot \eta_3$. Estimating this model using `lavaan` (Rosseel, 2012, version 6.5) with the ML estimator gives the results shown in Table 1. We also used `lavaan`’s option to mimic the EQS software to ensure the comparability of our results with those in Kwan and Chan (2011). The option `mimic = "EQS"` of the `sem()` function was used. The ML estimator with this option will be used in all examples in this tutorial. The table shows that within the original metric, the estimated unstandardized regression coefficients $\hat{\beta}_{42}$ and $\hat{\beta}_{43}$ differ considerably (0.081 vs. 0.257). Within the rescaled metric, the difference between both regression coefficients is smaller (0.081 vs. 0.026) than in the original metric. But for both the model in the original metric and the model in the rescaled metric, the difference of the estimated standardized regression coefficients $\hat{\beta}_{42}^S$ and $\hat{\beta}_{43}^S$ is the same (0.292 vs. 0.280). The example

¹the restrictions are set free yielding a χ^2 -distributed test statistic. We do not consider the score test in this tutorial.



Figure 1 ■ Model for the first example, the variances are indicated at the upper right of the variable symbols.



demonstrates that the standardized regression coefficients are invariant towards the metric of the variables. This is because the variance of η_2 is about ten times larger than the variance of η_3 (Kwan & Chan, 2011, p. 732). For the regression with the rescaled η_3 , the difference of the unstandardized regression coefficients is not as large as the difference in the original scale. The regression coefficient $\hat{\beta}_{43}$ for the model with the rescaled $\tilde{\eta}_3$ is (up to rounding) $\frac{1}{10}$ of the regression coefficient for the model in the original metric. The reason therefore is that the portion of the variance that is explained by $\tilde{\eta}_3$, or respectively η_3 , must be the same regardless of the metric. Because $\text{var}(\tilde{\eta}_3) = 10^2 \cdot \text{var}(\eta_3)$, we have for the portion of explained variance $(\frac{1}{10} \cdot \beta_{43})^2 \cdot \text{var}(\tilde{\eta}_3) = \frac{1}{100} \cdot \beta_{43}^2 \cdot 100 \cdot \text{var}(\eta_3) = \beta_{43}^2 \cdot \text{var}(\eta_3)$. This result shows the ML estimation is scale free, i.e. if a model is rescaled and a parameter is estimated for this model, then this parameter can be algebraically converted back to the original metric (Kline, 2016, p. 238). Moreover, the table demonstrates that for the rescaled model, the standardized regression coefficients are equal to those of the original model, which is easily verified by applying Equation (2).

As the example shows, the comparison of standardized regression coefficients is a possibility to examine the effects of explanatory variables. These remain the same in terms of their portion of explained variance, regardless of the metric of the explanatory variables. But the researcher should be aware that the tested hypothesis differs from the one when unstandardized regression coefficients are compared.

Although the comparison of standardized regression coefficients is a possibility to compare the effects of the explanatory variables on a dependent variable, the implementation is not straightforward because standardized coefficients are calculated after the estimation process (see the Mathematical Appendix for the standardization formulas). Directly calculating standardized coefficients re-

quires a model reparameterization (McDonald, Parker, & Ishizuka, 1993). Thus, to apply a LR test, the model must firstly be reparameterized and secondly, a restricted reparameterized model must be estimated which, thirdly, are compared. Therefore, Kwan and Chan developed a method to reparameterize a model by means of phantom and imaginary latent variables (Rindskopf, 1984). The original model and the reparameterized model are equivalent and the regression coefficients in the reparameterized model represent the standardized parameters in the original model. Thus, the – now standardized – regression coefficients in the reparameterized model can be tested by means of the LR test. Although this is a general approach, the model reparameterization is not a trivial task and is currently not implemented in standard structural equation software (Kwan & Chan, 2011). An alternative way to compare standardized regression coefficients is the Wald test. As the LR and the Wald test are asymptotically equivalent under the null hypothesis, they should provide the same results in the long run.

In this tutorial paper, we will demonstrate a method to test the equality of standardized regression parameters drawing on the Wald test for linear restrictions and the asymptotic covariance matrix of the standardized parameter estimates using the open source R package `lavaan`. The goal of this paper is educational: Firstly, we want to demonstrate how a typical textbook display of the Wald test can be used for comparing standardized parameters and secondly, we want to demonstrate how the Wald test can be programmed in the R language. The tutorial requires some knowledge of matrix algebra and statistical concepts like the asymptotic covariance matrix of the estimated parameters and the delta method to understand the calculations involved in the Wald tests (e.g., Casella & Berger, 2002; Greene, 2012; Fox, 2016). Additionally, the reader requires some knowledge of R and the `lavaan` package. In the following section, we recapitulate the ba-



Table 1 ■ Unstandardized and standardized estimates for Example 1.

Parameter	Original metric				Rescaled metric			
	Unstandardized		Standardized		Unstandardized		Standardized	
	Est.	SE	Est.	SE	Est.	SE	Est.	SE
$\hat{\beta}_{42}$	0.081	0.019	0.292	0.065	0.081	0.019	0.292	0.065
$\hat{\beta}_{43}$	0.257	0.062	0.280	0.065	0.026	0.006	0.280	0.065
$\hat{\psi}_{23}$	1.031	0.209	0.374	0.061	10.312	2.086	0.374	0.061
$\hat{\psi}_{22}$	9.069	0.909	1.000	0.000	9.069	0.909	1.000	0.000
$\hat{\psi}_{33}$	0.837	0.084	1.000	0.000	83.710	8.392	1.000	0.000
$\hat{\psi}_{44}$	0.550	0.055	0.776	0.052	0.550	0.055	0.776	0.052

sics of the Wald test, and show how the Wald test can be used to compare standardized regression coefficients. In the third section, we use the example from the introduction to develop R code to calculate the actual Wald test statistic given a fitted model. We apply the Wald test to an example with two restrictions in the fourth section and in the fifth section, we show how to apply the Wald test for standardized regression coefficients in a SEM involving latent variables. In the sixth section, we provide a convenience function that handles the Wald test for equality constraints of standardized regression coefficients with only requiring the constraints in the usual `lavaan` syntax and a fitted `lavaan` object. The last section discusses and concludes. Throughout the paper, we draw on the examples provided in Kwan and Chan (2011) and compare the results obtained from the Wald test to results from the LR test in Kwan and Chan (2011). In order to ensure the comparability with the examples in Kwan and Chan (2011), we use the same numbering of the variables in this tutorial as in the Kwan and Chan paper. Thus, the regression coefficients in this tutorial will have the same indices as in Kwan and Chan. A description of the variables used in the examples is provided in Appendix B. All R scripts for the examples in this tutorial are provided as supplementary material.

Wald test for standardized parameters

The method to compare standardized regression parameters shown in this section draws on the Wald test Wald, 1939 and its description provided in Greene (2012, p. 527-529). In this section, we firstly describe the Wald test and how it is applied to a set of linear restrictions. Because the Wald test is a general method to compare some model parameters, we describe the Wald test with regards to *unstandardized* parameters. Afterwards, we explain how *standardized parameters* are obtained from unstandardized parameters and then, thirdly, we show of the Wald test is adapted to compare the *standardized* parameters. The focus in this section is on an explication of the hypothesis behind and of the principles of the Wald test. Mathemati-

cal formulas are only provided as they are required for the later implementation in R and `lavaan`. In order to keep the tutorial self-contained, details of parameter standardization and of the Wald test are provided in the Mathematical Appendix.

In the example from the introduction, it was of substantial interest to statistically compare the two regression coefficients β_{42} and β_{43} . Stated otherwise the following hypothesis is to be tested:

$$H_0 : \beta_{42} = \beta_{43}. \tag{6}$$

Formally, the equation in the hypothesis can be rewritten in the form $\beta_{42} - \beta_{43} = 0$. This equation represents a restriction on the parameters. Using matrix algebra and introducing a number c , the equation becomes

$$(1 \quad -1) \cdot \begin{pmatrix} \beta_{42} \\ \beta_{43} \end{pmatrix} = c. \tag{7}$$

The two parameters β_{42} and β_{43} represent two parameters from the model. In this example, we have gathered these two parameters in a row vector. The restrictions are gathered in the column vector and it is easy to see that multiplying both vector yield the restriction $\beta_{42} - \beta_{43}$. The number c represents the restricted parameter values. The Hypothesis 6 is valid, when the restriction parameters values c only randomly deviates from zero.

In general, all parameters from a model are gathered in a vector which is the parameter vector θ . When there are p parameters in the model, the length of the parameter vector θ consequently is p . Now, we consider the general case in which we have a set of J linear restrictions. These can be written in form of a restriction matrix R . The number of rows of the restriction matrix R corresponds to the number of restrictions and the number of columns corresponds to the length of the parameter vector θ . Consequently, the dimension of R is $J \times p$ and the equation $R \cdot \theta = c$ yields the vector of restricted parameters values with length J .



In the general case, the hypothesis for a set of J linear restrictions becomes

$$H_0 : \mathbf{R} \cdot \boldsymbol{\theta} = \mathbf{0}. \tag{8}$$

The statistic W of the Wald test for this hypothesis is given by Greene (2012, p. 528) as

$$W = \left[\mathbf{R} \cdot \hat{\boldsymbol{\theta}} \right]^\top \cdot \left[\mathbf{R} \cdot \text{ACOV}(\hat{\boldsymbol{\theta}}) \cdot \mathbf{R}^\top \right]^{-1} \cdot \left[\mathbf{R} \cdot \hat{\boldsymbol{\theta}} \right] \tag{9}$$

with $\text{ACOV}(\hat{\boldsymbol{\theta}})$ being the asymptotic covariance matrix of the estimated parameters. Under H_0 , W is asymptotically χ^2 -distributed with J degrees of freedom. Stated otherwise, the number of df equals the number of rows in the restriction matrix.

The idea behind the Wald is that when the restrictions are valid, then the vector of restricted parameters values c only marginally deviates from the zero vector due to random sampling error. If the restriction are not valid, then the deviation of c from the zero vector is systematic which is reflected in a significant χ^2 -test statistic. This also entails that the Wald always refer to the total set of restrictions.

Up to now, we considered unstandardized parameters only. However, we want to test standardized parameters. The standardized parameters are easily obtained after the model has been estimated by means of the formulas provided, e.g., in Hayduk (1988, p. 181-183). These formulas transform the vector of estimated *unstandardized* parameters $\hat{\boldsymbol{\theta}}$ in the vector of estimated *standardized* parameters $\hat{\boldsymbol{\theta}}^S$. From Equation (9), we see that the asymptotic covariance matrix for the estimated standardized parameters is also needed to calculate a Wald Test. The asymptotic covariance matrix of the standardized estimated parameters $\text{ACOV}(\hat{\boldsymbol{\theta}}^S)$ can be obtained from the asymptotic covariance matrix of the estimated parameters $\text{ACOV}(\hat{\boldsymbol{\theta}})$ by using the delta method (Muthén, 2007; see also Greene, 2012, pp. 1083-1084). In `lavaan` there are functions that provide easy access to the required $\hat{\boldsymbol{\theta}}^S$ and $\text{ACOV}(\hat{\boldsymbol{\theta}}^S)$ so that no calculations by the user are needed.

Now, we adapt Equation (9) for the comparison of standardized regression coefficients. For the sake of convenience and to later ease the programming, we define the matrix

$$\mathbf{M} := \left[\mathbf{R} \cdot \text{ACOV}(\hat{\boldsymbol{\theta}}^S) \cdot \mathbf{R}^\top \right]. \tag{10}$$

Using the \mathbf{M} matrix, the expression for Wald test for standardized parameters becomes

$$W = \left[\mathbf{R} \cdot \hat{\boldsymbol{\theta}}^S \right]^\top \cdot \mathbf{M}^{-1} \cdot \left[\mathbf{R} \cdot \hat{\boldsymbol{\theta}}^S \right]. \tag{11}$$

In the following, we use Equation (11) to implement the Wald test for standardized regression parameters in

`lavaan`. The idea behind the implementation is simple: Firstly, the restriction matrix \mathbf{R} is determined by the hypothesis to be tested. Secondly, `lavaan` provides access to the estimated standardized parameters $\hat{\boldsymbol{\theta}}^S$ and the asymptotic covariance matrix of the estimated parameters $\text{ACOV}(\hat{\boldsymbol{\theta}}^S)$ so that W calculates according to Equation (11). Finally, the p -value of the Wald statistic is calculated using the build in `R` function for the χ^2 -distribution.

First example with one restriction

In this section, we use the PISA example from Kwan and Chan (2011) that was already given in the introduction. As a quick reminder, the substantial hypothesis in this regression model is whether the effects of parental educational level and childrens' home possession on childrens' educational resources at home are the same. The statistical null hypothesis is

$$H_0 : \beta_{42}^S = \beta_{43}^S \Leftrightarrow \beta_{42}^S - \beta_{43}^S = 0. \tag{12}$$

In this model, there are six parameters in total: two regression coefficients, the covariance between the exogenous variables, the two variances of the exogenous variables, and the error variance of the endogenous variable. The parameter vector of the standardized parameters with length $k = 6$ is

$$\boldsymbol{\theta}^S = (\beta_{42}^S, \beta_{43}^S, \psi_{23}^S, \psi_{22}^S, \psi_{33}^S, \psi_{44}^S). \tag{13}$$

The number of rows of the restriction matrix results from Hypothesis (12), in particular from the expression $\beta_{42}^S - \beta_{43}^S = 0$. Because there is one restriction, the restriction matrix has one row. When constructing a restriction matrix, it must be taken care that the sequence of the columns equals the sequence of the elements in the parameter vector. In particular, the entries representing the restrictions have to be placed in the correct columns. In the current example, 1 and -1 have to be placed in the first and second column. The restriction matrix for the example is

$$\mathbf{R} = [\beta_{42}^S - \beta_{43}^S = 0] \begin{pmatrix} [\beta_{42}^S & \beta_{43}^S & \psi_{23}^S & \psi_{22}^S & \psi_{33}^S & \psi_{44}^S] \\ 1 & -1 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

For educational purposes, we print the restriction before the row and the elements of the parameters above the restriction matrix in square brackets, so it is easy to see how to set up the restriction matrix. It is easy to check that $\mathbf{R} \cdot \boldsymbol{\theta}^S$ represents the parameter restrictions. Additionally, the example shows that to construct the restriction matrix, the sequence of the parameters in the parameter vector matters. If the sequence of the parameters in the column of the restriction matrix does not match the sequence in the parameter vector $\boldsymbol{\theta}^S$, the restrictions are not correctly implemented. In the example at hand, the



restriction matrix is already matched with the sequence of the parameters in θ^S but in practical applications, it is necessary to determine the sequence of parameters. One way to get the sequence of the parameters in `lavaan` is to use the `parTable()` function. This function return `lavaan`'s parameter table, which is a data frame representing to model. In the rows the parameter table indicates the parameters and the sequence of parameters is the same for all `lavaan` objects. With this information at hand, we can compute the Wald test. The calculation proceeds in pseudocode as described in Algorithm 1.

The first two requirements concern the asymptotic covariance matrix of the standardized parameters and the parameter vector of standardized parameters, which are provided by `lavaan` after a model has been fitted, whereas the third requirement concerns the restrictions matrix which is provided by the researcher. In the first step, the vector c with the restricted parameters is calculated. As mentioned in the section on the Wald test, when the restricted parameters deviate stronger from zero than explained by sampling variation, then the restriction is not valid. Although the calculation of the restricted parameters is not necessary, their calculation is informative from a didactic point of view. In the second step, we calculate a matrix M from Equation (10) and in the third step, we determine the inverse of M , which is needed in the next step for the calculation of the Wald test. The fourth step finally calculates the Wald statistic according to Equation (11). As the degrees of freedom of the Wald statistic are given by the number of rows of the restriction matrix R , this number is determined in step five by counting the number of rows. Finally, in the sixth and last step, the p -value for the Wald statistic W is calculated.

To conduct the Wald test, a fitted `lavaan` object containing the results for the unrestricted model is required. For the following R code, we assume that the object `fit` contains the fitted model (see the supplementary material for the model syntax). As mentioned above, the sequence of parameters in the restriction matrix must match the sequence of parameters represented in `lavaan`. Therefore, we print the parameter table with the following command shown in Listing 1.

The parameters are encoded in the second, third, and fourth column of the parameter table. The columns indicate the left hand side, the operator, and the right hand side of a parameter specification `lavaan`. For instance, the regression coefficient β_{42} is in the first row indicated by its `lavaan` syntax `eta4~eta2`. Thus, the sequence of parameters in `lavaan` is the same as shown in Equation 13. Now, we can set up the R code for the Wald test as given in Listing 2.

In the first line of the code the covariance matrix of the standardized estimated parameters is extracted using the `inspect()` function from the `lavaan` package. With the argument `"vcov.std.all"`, the covariance matrix of the standardized estimated parameters is requested. The requested covariance matrix is formatted as a `lavaan.matrix.symmetric` object, R's `unclass()` function converts this into a R table. In the second line, the vector of standardized estimated parameters is extracted by means of `lavaan`'s `standardizedSolution()` function. This function returns a data frame from which we only need the column containing the standardized parameters. Therefore, we add `["est.std"]` behind the call `standardizedSolution()` function. Afterwards, we also use the `matrix()` function from R to convert the extracted parameter vector to a matrix, which is later needed for the upcoming matrix multiplications. The restriction matrix is set up in the third line. In lines four to nine, the Wald statistic W and its p -value are calculated with $df = 1$. In the fourth line, the value of the restricted parameters is calculated. Note, that `%%` indicates matrix multiplication. In the fifth line, the matrix M is calculated. As in the Equation (11) the inverse matrix M^{-1} is needed, the inverse matrix is calculated in the sixth line with the `solve()` function; providing a matrix to the `solve()` function provides the inverse of this matrix. In line seven, the Wald statistic is calculated, the function `t()` indicates the transpose of a matrix. The degrees of freedom are determined in line eight by counting the rows of the restriction matrix with the `nrow()` function. Finally, in the ninth line, the p -value is calculated with the `pchisq()` function. Running the code yields the following results for the constraint parameter:

```
> c
      [,1]
[1,] 0.01207366
```

The value of the restricted parameters deviates only marginally from zero as revealed by the non-significant Wald test statistic:

```
> W
      [,1]
[1,] 0.0113777
> p
      [,1]
[1,] 0.9150537
```

Thus, the restriction is not detrimental and the result is almost identical to the results presented in Kwan and Chan (2011, p. 732, Table 2). Their χ^2 -difference test, which corresponds to the LR test using the reparameterization approach, is $\Delta\chi^2 = 0.011$ with $\Delta df = 1$ and $p = .916$.



Algorithm 1 Calculate Wald test

Require: $\text{ACOV}(\hat{\theta}^S)$
Require: θ^S
Require: R

- 1: $c \leftarrow R \cdot \theta$
- 2: $M \leftarrow R \cdot \text{ACOV}(\hat{\theta}^S) \cdot \text{transpose}(R)$
- 3: $M^{-1} \leftarrow \text{determine-inverse}(M)$
- 4: $W \leftarrow \text{transpose}(R \cdot \theta^S) \cdot M^{-1} \cdot R \cdot \theta^S$
- 5: $df \leftarrow \text{count-rows}(R)$
- 6: $p \leftarrow 1 - P_{\chi^2}(W, df)$

Listing 1 ■ Command to obtain the parameter table.

```
> parTable(fit)
  id lhs op rhs user block group free ustart exo label plabel start est se
1  1 eta4 ~ eta2  1  1  1  1  NA  0  b42  .p1. 0.000 0.081 0.019
2  2 eta4 ~ eta3  1  1  1  2  NA  0  b43  .p2. 0.000 0.257 0.062
3  3 eta2 ~~ eta3  1  1  1  3  NA  0  cov23 .p3. 0.000 1.031 0.209
4  4 eta2 ~~ eta2  1  1  1  4  NA  0  var2  .p4. 4.535 9.069 0.909
5  5 eta3 ~~ eta3  1  1  1  5  NA  0  var3  .p5. 0.419 0.837 0.084
6  6 eta4 ~~ eta4  1  1  1  6  NA  0  resvar4 .p6. 0.354 0.550 0.055
```

Second example with two restrictions

The next example, for which we want to compare standardized regression coefficients, is Example 1 from Kwan and Chan (2011, p. 736). In this example, we use four out of the five variables found in the covariance matrix in Appendix B.1. We use the same numbering of the variables as in Kwan and Chan so that the regression coefficients will have the same indices as in their paper. In this second example, we again consider a regression model but with three exogenous variables, it is depicted in Figure 2. The example draws on the same data set and covariance matrix as the previous example.

In this example, the three explanatory variables are parental occupational status (η_1), parental educational level (η_2), and home possessions (η_3). The dependent variable is reading ability (η_5). The results for the estimated model are shown in Table 2.

It is of substantial interest, if the effects of the explanatory variables on the dependent variable are equal. Thus, we have the following hypothesis concerning the equality of the standardized regression coefficients β_{51}^S , β_{52}^S , and β_{53}^S :

$$H_0 := \left\{ \begin{matrix} \beta_{51}^S = \beta_{52}^S \\ \beta_{52}^S = \beta_{53}^S \end{matrix} \right\} \Leftrightarrow \left\{ \begin{matrix} \beta_{51}^S - \beta_{52}^S = 0 \\ \beta_{52}^S - \beta_{53}^S = 0 \end{matrix} \right\}. \quad (15)$$

In this model, we have ten parameters and the parameter vector of the standardized parameters with length $k = 10$ is

$$\theta^S = (\beta_{51}^S, \beta_{52}^S, \beta_{53}^S, \psi_{12}^S, \psi_{13}^S, \psi_{23}^S, \psi_{11}^S, \psi_{22}^S, \psi_{33}^S, \psi_{55}^S). \quad (16)$$

As we have two restrictions, the restriction matrix has two rows and ten columns, it is

$$R = \begin{bmatrix} \beta_{51}^S - \beta_{52}^S = 0 \\ \beta_{52}^S - \beta_{53}^S = 0 \end{bmatrix} \begin{pmatrix} \beta_{51}^S & \beta_{52}^S & \beta_{53}^S & \psi_{12}^S & \psi_{13}^S & \psi_{23}^S & \psi_{11}^S & \psi_{22}^S & \psi_{33}^S & \psi_{55}^S \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \quad (17)$$



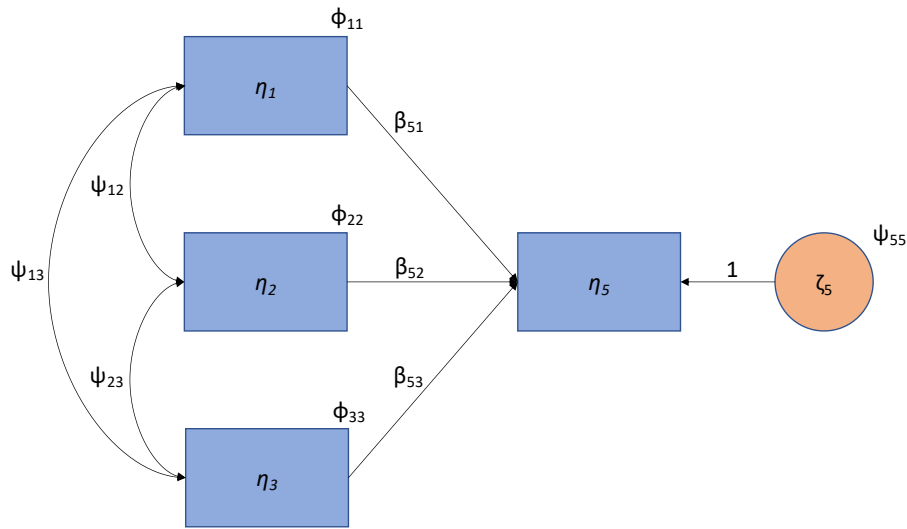
Listing 2 ■ Code for the Wald test.

```

1 SVCOV <- unclass(inspect(fit, "vcov.std.all"))
2 theta <- matrix(standardizedSolution(fit)[,"est.std"], 6, 1)
3 R <- matrix(c(1,-1,0,0,0,0), 1,6)
4 c <- R %*% theta
5 M <- R %*% SVCOV %*% t(R)
6 M.inv <- solve(M)
7 W <- t(R %*% theta) %*% M.inv %*% R %*% theta
8 df <- nrow(r)
9 p <- 1-pchisq(W, df)

```

Figure 2 ■ Model for the second example, the variances are indicated at the upper right of the variable symbols.



As in the first example, we put the restriction before and the elements of the parameter vector above the restriction matrix because in this example, attention must be paid to the construction of the restriction matrix. The first hypothesis about the equality of β_{51}^S and β_{52}^S refers to the first and second element of the parameter vector θ^S and therefore, the equality constraints are encoded in the first and second column of the first row of the restriction matrix. The second hypothesis about the equality of β_{52}^S and β_{53}^S refers to the second and third parameter in the parameter vector θ^S . Therefore, the equality constraints are encoded in the second and third column of the second row of the restriction matrix. As the restriction matrix has two rows, there are two degrees of freedom for the Wald test.

Now, to perform the Wald test, the sequence of the parameters in lavaan has to be determined using the parTable() function as shown in the first example. Afterwards, the Wald test can be calculated with the same

R code as in the previous example. This yields for the restricted parameters:

```

> c
      [,1]
[1,] 0.07158408
[2,] -0.17025339

```

This example shows that there are two restricted parameters yielding a restricted parameter vector of length 2 and also demonstrates that the Wald test always refers to the entire set of restrictions. The results for the Wald statistic and its p -value are:

```

> W
      [,1]
[1,] 2.098924
> p
      [,1]
[1,] 0.3501261

```




Table 2 ■ Unstandardized and standardized estimates for Example 2.

Parameter	Unstandardized		Standardized	
	Est.	SE	Est.	SE
$\hat{\beta}_{51}$	0.626	0.470	0.121	0.090
$\hat{\beta}_{52}$	1.273	2.369	0.049	0.091
$\hat{\beta}_{53}$	18.712	6.261	0.219	0.072
$\hat{\psi}_{12}$	29.923	3.848	0.661	0.040
$\hat{\psi}_{13}$	4.881	1.035	0.355	0.062
$\hat{\psi}_{23}$	1.031	0.209	0.374	0.061
$\hat{\psi}_{11}$	226.258	22.683	1.000	0.000
$\hat{\psi}_{22}$	9.069	0.909	1.000	0.000
$\hat{\psi}_{33}$	0.837	0.084	1.000	0.000
$\hat{\psi}_{55}$	5481.009	549.476	0.900	0.040

The results are close to the results of the LR test using the reparameterization approach in Kwan and Chan (2011, p. 738, Table 4). Their χ^2 -difference test is $\Delta\chi^2 = 2.070$ with $\Delta df = 2$ and $p = .355$.

Third example involving latent variables

Our third example involves latent variables and corresponds to Example 3 in Kwan and Chan (2011, p. 738). The example draws on a data set with $N = 6407$ participants provided in Schoon and Parsons (2002). The model involves three latent variables $\eta_1, \eta_2,$ and η_3 with each latent variable being measured by two manifest variables. It is depicted in Figure 3 and the covariance matrix is given in Appendix B.2.

The latent variable η_1 , educational achievement, is measured by examination score (y_1) and highest qualification (y_2), the latent variable η_2 , teenage aspiration, is measured by job aspiration (y_3) and educational aspiration (y_4), and the latent variable η_3 , occupational attainment, is measured by Goldthorpe (y_5) and RGSC (y_6) (see Appendix B for explanations of the manifest variables).

As this example involves latent variables, the latent variables have to be scaled so that the model parameters can be estimated (Kline, 2016, p. 199). In this section, we use Fixed Marker scaling, which restricts one loading to 1. The de facto standard is to use the first indicator of a

latent variable as marker. Thus, for models involving latent variables, there are two scaling issues. Firstly, there is the arbitrary scaling of the manifest variables in the sense of a possible linear transformation as described in the introduction. Secondly, there is the issue of scaling the latent variable. Both scaling issues cause the estimated regression coefficients to depend on the given metric of the manifest variables and the scaling method used to scale the latent variable. Fortunately, the standardized regression coefficients are also invariant towards the particular applied scaling method for the latent variables (see Klößner & Klopp, 2019; Klopp & Klößner, 2020).

Estimating the model applying the Fixed Marker scaling with the first indicator as marker yields a well-fitting model, $\chi^2(df = 6) = 1.199, p = .977, CFI = 1.000, RMSEA = 0.000, p_{close} = 1.000,$ and $SRMR = 0.002$. The estimated parameters are shown in Table 3.

The substantial hypothesis of interest is if the effects of the latent variables η_1 and η_2 on the latent variable η_3 are equal. In this case, the hypothesis is analogous to the hypothesis in the first example. The actual hypothesis is

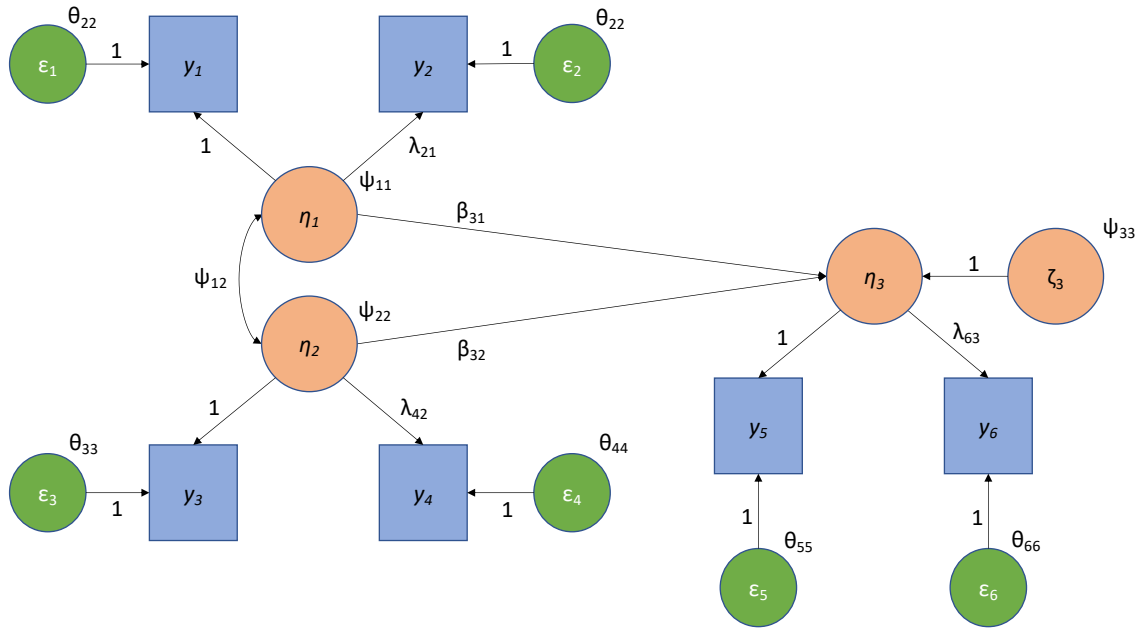
$$H_0 : \beta_{31}^S = \beta_{32}^S \Leftrightarrow \beta_{31}^S - \beta_{32}^S = 0. \tag{18}$$

The parameter vector of the standardized parameters with length $k = 18$ is

$$\theta^S = (\lambda_{11}^S, \lambda_{21}^S, \lambda_{32}^S, \lambda_{42}^S, \lambda_{53}^S, \lambda_{63}^S, \beta_{31}^S, \beta_{32}^S, \psi_{11}^S, \psi_{22}^S, \psi_{33}^S, \psi_{12}^S, \theta_{11}^S, \theta_{22}^S, \theta_{33}^S, \theta_{44}^S, \theta_{55}^S, \theta_{66}^S). \tag{19}$$



Figure 3 ■ Model for the third example, the variances are indicated at the upper right of the variable symbols.



In this parameter vector, we have to be aware that the estimated parameters $\hat{\lambda}_{11}$, $\hat{\lambda}_{32}$, and $\hat{\lambda}_{53}$ are restricted to 1 so that they are not freely estimated, although these restricted parameters can be standardized. This can be seen in Table 3. Because these parameters are not freely estimated, they are not contained in the asymptotic covariance matrix. Thus, the dimension of $ACOV(\hat{\theta}^S)$ is 15×15 .

Using the `standardizedSolution()` function, however, returns a vector with length 18. Thus, to be conformable for matrix multiplication, we have to remove the fixed parameters from this vector. The sequence of the parameters is as displayed in Equation (19), so that the first, third and fourth element of the vector must be removed. This “reduced” vector is

$$\theta_{red}^S = (\lambda_{21}^S, \lambda_{42}^S, \lambda_{63}^S, \beta_{31}^S, \beta_{32}^S, \psi_{11}^S, \psi_{22}^S, \psi_{33}^S, \psi_{12}^S, \theta_{11}^S, \theta_{22}^S, \theta_{33}^S, \theta_{44}^S, \theta_{55}^S, \theta_{66}^S). \tag{20}$$

Thus, as in the previous examples, we need to determine the sequence of parameters in `lavaan` and additionally determine which parameter is fixed and which is free. Again, this can be done by using the `parTable()` function. Calling the function yield the (truncated) Listing 3.

The parameter table contains the column `free` which indicates if a parameter is fixed or free. For a fixed parameter, the value in the `free` column equals zero, whereas for a free parameter the value in the `free` column is different from zero. As can be seen in the parameter table, the parameters for each first loading of the manifest variable on each latent is fixed, which is straightforward as the Fixed Marker scaling with the first indicator as marker was

used. Thus, to get this reduced parameter we have determine which parameters are free. To achieve this, we have to alter the R code that provides the parameter vector:

```
1 theta <- matrix(standardizedSolution
2                 (fit)[,"est.std"], 18, 1)
3 fixed <- which(unclass(parTable
4                 (fit))$free %in% c(0))
5 theta <- theta[-fixed]
```

In the first and second lines, the parameter vector of the standardized parameters is extracted as above. The third and fourth lines determine which parameters in the parameter table are fixed and which parameters are free



Table 3 ■ Unstandardized and standardized estimates for Example 3.

Parameter	Unstandardized		Standardized	
	Est.	SE	Est.	SE
$\hat{\lambda}_{11}$	1.000	0.000	0.898	0.010
$\hat{\lambda}_{21}$	0.028	0.001	0.623	0.010
$\hat{\lambda}_{32}$	1.000	0.000	0.488	0.012
$\hat{\lambda}_{42}$	3.250	0.113	0.859	0.013
$\hat{\lambda}_{53}$	1.000	0.000	0.843	0.009
$\hat{\lambda}_{63}$	0.084	0.002	0.925	0.010
$\hat{\beta}_{31}$	0.192	0.024	0.212	0.025
$\hat{\beta}_{32}$	16.333	1.570	0.278	0.026
$\hat{\psi}_{11}$	212.209	6.427	1.000	0.000
$\hat{\psi}_{22}$	0.050	0.003	1.000	0.000
$\hat{\psi}_{33}$	138.165	4.302	0.793	0.011
$\hat{\psi}_{12}$	2.332	0.095	0.713	0.014
$\hat{\theta}_{11}$	51.204	4.613	0.194	0.018
$\hat{\theta}_{22}$	0.267	0.006	0.611	0.013
$\hat{\theta}_{33}$	0.162	0.003	0.762	0.012
$\hat{\theta}_{44}$	0.189	0.016	0.262	0.022
$\hat{\theta}_{55}$	70.797	3.798	0.289	0.016
$\hat{\theta}_{66}$	0.208	0.026	0.144	0.018

using the `parTable()` function. The `unclass()` function converts the data frame to a list of vectors from which the `free` vector is selected by means of the `$` operator. The `%in%` operator determines the position of the fixed parameters by determining the position of the zeros in the vector. Finally, in the fifth line, the fixed parameters are removed from the vector of standardized parameters extracted in the first line.

Now the restriction matrix must be set up. As is the previous examples, the use of the `parTable()` function gives the sequence parameters but in this example only for the free parameters. As there is one restriction, it has one row. The restrictions for the equality of the standardized regression coefficients must be in the fourth and fifth column according to their position in the reduced parameter vector. The restriction matrix is

$$R = (0 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0)_{(21)}$$

Running the R code yields for the restricted parameter:

```
> c
      [,1]
[1,] -0.06577379
```

Again, the restricted parameter deviates only marginally from zero. The Wald statistic and its *p*-value are:

```
> W
      [,1]
[1,] 1.756541
> p
      [,1]
[1,] 0.1850566
```

This result is also in line with the LR test in Kwan and Chan (2011, p. 743, Table 8). Their χ^2 -difference test is $\Delta\chi^2 = 1.782$ with $\Delta df = 1$ and $p = .182$.

A convenience function

As the third example shows, it may be necessary to manually alter the R code to calculate the Wald statistic and moreover, it may be clumsy to set up the restriction matrix by hand. Therefore, we adapted the `lavTestWald()` function from the `lavaan` package. The supplementary material (found on the journal’s web site) contains the R code of this function, which is of course in large parts identical to the code found in the `lavaan` package². All changes made to the original function are marked in the R

²The `lavaan` source code is on github, <https://github.com/yrosseel/lavaan/>, and the original `lavTestWald` function is in the file “`lav_test_Wald.R`”.



Listing 3 ■ Output to determine fixed and free parameters.

```
> parTable(fit)
  id lhs op  rhs user block group free  ustart  exo  label plabel  start    est    se
1  1 eta1 =~  y1   1    1    1    0    1    0    .p1.  1.000  1.000  0.000
2  2 eta1 =~  y2   1    1    1    1    NA   0    .p2.  0.045  0.028  0.001
3  3 eta2 =~  y3   1    1    1    0    1    0    .p3.  1.000  1.000  0.000
4  4 eta2 =~  y4   1    1    1    2    NA   0    .p4.  0.351  3.250  0.113
5  5 eta3 =~  y5   1    1    1    0    1    0    .p5.  1.000  1.000  0.000
6  6 eta3 =~  y6   1    1    1    3    NA   0    .p6.  0.119  0.084  0.002
7  7 eta3 ~ eta1  1    1    1    4    NA   0    b31  .p7.  0.000  0.192  0.024
8  8 eta3 ~ eta2  1    1    1    5    NA   0    b32  .p8.  0.000 16.333  1.570
...
```

code and draw on the programming presented so far. The R code for the adapted function is in the supplemental material. The adapted function is called `lwt()` and the syntax for the function call is

```
lwt(fit, constraints = NULL, verbose =
    FALSE, std = FALSE)
```

where `fit` indicates a fitted lavaan object and `constraints` contains a string with the restrictions. The logical argument `verbose` indicates if the restriction matrix and the estimated restricted values are printed. Besides the arguments of `lavTestWald()`, the `lwt()` function adds the additional logical argument `std`. When `std = TRUE`, the Wald test will be calculated for the standardized parameters. The standard setting is `std = FALSE`.

The key of using the `lwt()` function (and also the `lavTestWald` function) is the use of parameter labels. Parameter labels allow to reference parameters in lavaan directly (see Rosseel, 2020, p. 13). We will now explain the use of the `lwt()` function with the second example. In this example, we use the parameter label `b51` for the regression coefficient β_{51} and the label `b52` for the regression coefficient β_{51} . Now, the restrictions have to be set up in the way as described in the lavaan documentation. For this example, the restrictions are implemented by

```
constr <- 'b51 == b52
          b52 == b53'
```

and lavaan will automatically convert these restrictions to a restriction matrix. Each restriction must be in a separate line (but see the lavaan documentation for an alternative way). Now, the call of the `lwt()` function is

```
lwt(fit, constraints = constr, std =
    TRUE, verbose = TRUE)
```

where `verbose = TRUE` indicates to print the restric-

tion matrix and the restricted theta values should be printed. Calling the function gives the following results:

```
Restriction matrix (jacobian):
[,1] [,2] [,3] [,4] [,5] [,6] [,7] [,8] [,9] [,10]
  1  -1  0  0  0  0  0  0  0  0
  0  1  -1  0  0  0  0  0  0  0

Restricted theta values:
[1] 0.07158408 -0.17025339

$stat
[1] 2.098924

$df
[1] 2

$p.value
[1] 0.3501261
```

As can be seen, the result of the `lwt()` functions equals the results of the manual calculation above. In addition, the `lwt()` as well as the `lavTestWald()` function print the information regarding which kind of standard error was calculated during the estimation of the model. As this information is not of relevance here, it was omitted from the output, but running the code in the supplementary material will print this, too.

Discussion

Up to here, we demonstrated how the Wald test is applied to compare standardized regression coefficients and how this is programmed in R. The method based on the Wald test is simple to implement and does not require estimating an unrestricted and restricted model to conduct a LR test. The developed convenience function also enables applied researchers to quickly conduct a Wald test to test hypotheses about parameter restrictions. Moreover, the results of the Wald test introduced here and the results of the LR test procedure in Kwan and Chan (2011) are very close or yield the same conclusion about the hypothesis, which is expected when sample size is large, because both



tests are asymptotically equivalent under the null hypothesis (Greene, 2012, p. 526).

Before discussing some limitations of the Wald test approach, we want to mention some issues regarding the use of standardized data. In structural equation modeling, the use of standardized data would consist in using a correlation matrix instead a covariance matrix to estimate a model. However, using a correlation matrix to estimate a model may work in some circumstances but is not without problems in general when the ML estimator. Using a correlation matrix may yield the lose of the scale freeness and scale invariance of the ML estimator (Kline, 2016, p. 238). Moreover, using a correlation matrix is also problematic when latent variables are involved. Latent variable need to be scaled so that the model is identified. For a particular scaling method, the estimates may differ if either a correlation or a covariance matrix is analyzed or the estimates may differ depending on the scaling method when correlation matrices are analyzed. However, standardizing estimates is invariant towards the choice of a particular scaling method (see Klopp & Klößner, 2020). In order to correctly fit a model using a correlation matrix, the constraint optimization method can be used. Constraint optimization consists in imposing non-linear constraints on some model parameters (Kline, 2016, p. 254). Another possibility to directly estimate standardized parameters is model reparameterization, an instance of which is the method using phantom and imaginary latent variables develop by Kwan and Chan (2011). However, as mentioned in the introduction such a reparameterization is tedious and therefore, the approach using Wald tests was demonstrated.

Regarding the limitations, we only considered one type of standardization, in which both – explanatory and dependent – variables where standardized. There may be situations in which this may be inappropriate, e.g., when an explanatory variable is a dummy-coded variable (see Fox, 2016, pp. 149-150; and Newman & Browner, 1991, pp. 383-384). For a dummy-coded variable, the regression coefficients indicates the difference between a certain group and the reference group. According to Fox, this interpretation is lost when a regression coefficient for a dummy explanatory variable is standardized and moreover, as a 0/1 coded dummy variable cannot increase one standard deviation, the usual interpretation of a standardized regression coefficient does also not apply. Similar objections apply to regression coefficients of the interaction of dummy and continuous explanatory variable. These also should not be standardized, because they cannot change independently of the main effect regression coefficients. The Wald test has also the drawback that its results are not invariant towards non-linear reparameterizations of the restrictions (see Greene, 2012, p. 529). However, this is not relevant in

the context of this tutorial and we only want to mention this issue for the sake of completeness.

We also restricted our considerations to hypotheses of the form $\mathbf{R} \cdot \boldsymbol{\theta} = \mathbf{0}$, but an extension to hypotheses in the form $\mathbf{R} \cdot \boldsymbol{\theta} = \mathbf{q}$ is possible (see the Mathematical Appendix for details). We also want to mention that although we only used the ML estimator in all examples, the application of the Wald test is not limited to the ML estimator. As the Wald test is a general type of statistical test, it can be applied to test restrictions of parameters estimated under any estimator (see Greene, 2012, chapter 5.6) for an application with linear models and the least squares estimator). Moreover, all examples were variations of multiple regression models, but the Wald test approach certainly applies to more complex models.

Although the use of standardized regression coefficients was featured in this tutorial, there are several arguments against the use of standardized coefficients. Greenland, Schlesselman, and Criqui (1986) as well as Greenland, Maclure, Schlesselman, Poole, and Morgenstern (1991) argue that comparing standardized coefficients confounds effects with the explanatory and dependent variables' standard deviation. As already mentioned in the introduction, the interpretation of the hypothesis when comparing unstandardized regression coefficients in Equation (1) is fundamentally different from the hypothesis when comparing standardized regression coefficients in Equation (4), and the condition from Equation (5) when they are equivalent is hardly ever met. The interpretation of an unstandardized regression coefficient is that when the explanatory variable changes one unit, then the dependent variable changes β_{ki} units (at least in models without latent variables, in models with latent variables this interpretation is more complicated, see Klopp & Klößner, 2020). In contrast, the interpretation of a standardized regression coefficient is that when the explanatory variable changes one standard deviation, the dependent variables changes β_{ki}^S standard deviations. Thus, the interpretation of hypotheses about standardized regression coefficients depends on the variables' standard deviations. Greenland et al. (1991) argue that the standard deviations of the involved variables depend on the study design. For instance, the application of criteria determining which cases are included in a study and which are excluded, may alter a variable's standard deviation. Another example for a potential factor influencing the standard deviation of a variable is the measurement design, e.g., how many categories in a rating scale are provided to the respondents. In such cases, the effect would be confounded with features of the study design and as a consequence, the size of the standard coefficients may depend on such features. Additionally, interpreting standard deviations may be problematic when the



involved variables are non-normally distributed.

Fox (2016, pp. 100-102) hints that standardized regression coefficients are of limited use when comparing the relative impact of incommensurable explanatory variables. Variables are incommensurable when they are measured in different units. For example, income is regressed on education and occupational experience (Fox, 2016, p. 101). When both explanatory variables are measured in years, they are commensurable. But when points are assigned to measure both variables they are incommensurable and their regression coefficients are in turn standardized to allow comparison. But the standardization by means of the variables' standard deviations itself is arbitrary. Fox (2016, p. 102) concludes that "standardized regression coefficients permit a limited comparison of the relative impact of incommensurable explanatory variables." Thus, care should be taken if reporting or comparing standardized regression coefficients is adequate (see also Baguley, 2009, for a general discussion regarding simple or standardized effect sizes).

Furthermore, drawing on the standard deviation to interpret the standardized regression coefficient has the drawback that it provides no information about the strength of an effect. In consequence, we want to argue that the comparison of standardized coefficients should refer to the portions of variance explained that two or more explanatory variables transfer to the dependent variable. This is because in general, the explanatory variables covary (see also Kim & Ferree, 1981). For instance, in the first example, the variables η_2 and η_3 covary with $\text{cov}(\eta_2, \eta_3)$. Using unstandardized model parameters, according to the formula for the variance of linear transformations of random variables, the total variance of the dependent variable therefore is $\text{var}(\eta_{44}) = \beta_{42}^2 \psi_{22} + \beta_{43}^2 \psi_{33} + 2\beta_{42}\beta_{43}\psi_{23} + \psi_{44}$. The effects of η_2 and η_3 on η_4 cannot be told apart because of the covariation between η_2 and η_3 . Consequently, the interpretation of the equality of two standardized regression coefficients has to refer to the portion of variance each explanatory variable contributes to the total variance of η_4 . Because using totally standardized parameters is tantamount to use standardized variables for estimation (see Kim & Ferree, 1981), i.e., $\psi_{22}^S = \psi_{33}^S = 1$, the expression for the total variance of η_4 becomes $\text{var}(\eta_{44}) = (\beta_{42}^S)^2 + (\beta_{43}^S)^2 + 2\beta_{42}^S\beta_{43}^S\psi_{23}^S + \psi_{44}^S = 1$. In the first example, the Wald test indicates that the hypothesis of the equality of the standardized regression coefficients holds, and the interpretation is: Both explanatory variables contribute equally to the total variance of the dependent variable. This interpretation also applies in cases in which there is no covariation between explanatory variables, even though there is no covariance term in such a situation. Additionally, the expression for the total vari-

ance also shows that for the portion of variance interpretation the sign of the relation vanishes because the standardized regression coefficients are squared. Thus, in this kind of interpretation, information about the direction of the relation is seemingly lost. But the direction of a relation is a substantial and not a statistical issue. For example, when the ML estimator is used, due to being scale free, the sign of the relation between η_2 and η_4 can be reversed by applying the linear transformation $\tilde{\eta}_2 = -1 \cdot \eta_2$ without changing the absolute value of the (un-)standardized regression coefficient. Thus, the interpretation in terms of portion of variance explained is in accordance with substantial interests.

In this tutorial, we provided an approach to test the equality of standardized regression coefficients in SEMs using the Wald test for standardized parameters and demonstrated how to implement this approach using R and lavaan. Although the implementation is easy with some background knowledge, applied researchers should be aware of the different interpretation of unstandardized and standardized coefficients and the possibility, that the size of the standard deviations depends on design features of the study and other factors.

Author's note

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A Mathematical Appendix

A.1 The LISREL all-Y notation

In the following, we introduce the LISREL all-Y notation. An advantage of this notation is that it corresponds to the way in which lavaan represents the parameter matrices of structural equation models. For p latent variables η , r endogenous residual errors ζ , m manifest variables y measuring the latent variables, and m manifest residual errors ϵ , the LISREL all-y notation uses the following model matrices: Λ , B , Ψ , Θ_ϵ (see Newsom, 2015, pp. 383–386). The parameter vectors and matrices are described in the tables below. A peculiarity of the LISREL all-y notation is that exo- and endogenous variables are not explicitly distinguished. Thus, for the vector of latent variables η , there are q exogenous and r endogenous variables with $p = q + r$. The measurement model is given by

$$y = \Lambda \cdot \eta + \Theta_\epsilon \quad (22)$$

and the structural model is given by

$$\eta = B \cdot \eta + \zeta. \quad (23)$$

Under the usual assumptions (e.g., Bollen, 1989), the covariance matrix for the manifest variables y is

$$\Sigma = \Lambda \cdot (I - B)^{-1} \cdot \Psi \cdot [(I - B)^{-1}]^\top \cdot \Lambda^\top + \Theta_\epsilon \quad (24)$$



Table 4 ■ Parameter vectors in the LISREL all-y notation.

Vector	Length	Description
\mathbf{y}	m	Manifest variables measuring the latent variables
ϵ	m	Manifest residual errors
$\boldsymbol{\eta}$	p	Latent variables
ζ	p	Latent residual errors for q endogenous latent variables, the elements for the r exogenous variables are zero

Table 5 ■ Parameter matrices in the LISREL all-y notation.

Matrix	Dimension	Description
$\mathbf{\Lambda}$	$m \times p$	Loadings of the manifest variables on the latent variables
\mathbf{B}	$p \times p$	Path coefficients
$\mathbf{\Psi}$	$p \times p$	Variances of the exogenous variables, or residual variances of the endogenous variables, or covariances for the exogenous variables, or covariance of the residual variances for the endogenous variables
Θ_ϵ	$m \times m$	Variances of the measurement error of the manifest variables

and the covariance matrix of the latent variables $\boldsymbol{\eta}$ is

$$\text{cov}(\boldsymbol{\eta}) = (\mathbf{I} - \mathbf{B})^{-1} \cdot \mathbf{\Psi} \cdot [(\mathbf{I} - \mathbf{B})^{-1}]^T. \tag{25}$$

If there are no latent variables, then the $\mathbf{\Lambda}$ and Θ_ϵ matrices are omitted and $\Sigma = \text{cov}(\boldsymbol{\eta})$, or equivalently, the $\mathbf{\Lambda}$ is set to the identity matrix and the Θ_ϵ to the zero matrix. A peculiarity of the LISREL all-y notation is that for exogenous variables, the model parameter ψ_{ii} refers to the variance, i.e., $\text{var}(\eta_i) = \psi_{ii}$, whereas for endogenous variables, the model parameter ψ_{ii} refers to the residual error variance.

A.2 Standardization formulas

Standardization is done after a model is estimated. The standardized estimated parameters $\hat{\boldsymbol{\theta}}^S$ are easily obtained by using the formulas provided in Hayduk (1988, p. 181-183). Standardization requires a diagonal $p \times p$ matrix \mathbf{S}_η which draws the covariance matrix of the (latent) variables (see Equation (25)) in a SEM:

$$\mathbf{S}_\eta := \begin{pmatrix} \text{var}(\eta_1)^{1/2} & 0 & \dots & 0 \\ 0 & \text{var}(\eta_2)^{1/2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \text{var}(\eta_p)^{1/2} \end{pmatrix} := \text{dg}(\text{cov}(\boldsymbol{\eta}))^{1/2}, \tag{26}$$

(see Magnus & Neudecker, 2007, p. 6) for the dg notation. The standardized loadings are:

$$\mathbf{\Lambda}_S = \mathbf{\Lambda} \cdot \mathbf{S}_\eta \tag{27}$$

The standardized regression coefficients are:

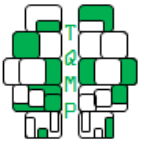
$$\mathbf{B}_S = \mathbf{S}_\eta^{-1} \cdot \mathbf{B} \cdot \mathbf{S}_\eta \tag{28}$$

The standardized latent (residual) covariance matrix is:

$$\mathbf{\Psi}_S = \mathbf{S}_\eta^{-1} \cdot \mathbf{\Psi} \cdot \mathbf{S}_\eta^{-1} \tag{29}$$

The correlation matrix of $\boldsymbol{\eta}$ is:

$$\text{cor}(\boldsymbol{\eta}) = \mathbf{S}_\eta^{-1} \cdot \text{cov}(\boldsymbol{\eta}) \cdot \mathbf{S}_\eta^{-1} \tag{30}$$



A.3 The Wald test

The method to compare standardized regression in his tutorial is the Wald test. In this section, following the reasoning provided in Greene (2012, p. 527-529), we firstly describe the Wald test and how it is applied to a set of linear restrictions and secondly, show how the Wald test for a set of linear restrictions is applied to compare standardized regression parameters.

The Wald test handles specific constraints on some parameters in a model. In the following, θ denotes the parameter vector of a SEM with length k and $\hat{\theta}$ denotes the estimated parameters. Then, the hypothesis of the equality of some parameters can be written as a set of J restrictions:

$$H_0 : c(\theta) = q \Leftrightarrow c(\theta) - q = \mathbf{0}. \tag{31}$$

The idea behind the Wald test is that if the restrictions hold true, then $\hat{\theta}$ should satisfy them. In this case, the vector with the restricted parameters $c(\hat{\theta}) - q$ should be approximately zero. In case the restrictions do not hold true, then the deviation of $c(\hat{\theta}) - q$ from zero should be more than explained by sampling variance.

According to Greene (2012, p. 528, Theorem 14.6), the limiting distribution of the Wald test is

$$W = [c(\hat{\theta}) - q]^\top \cdot \text{ACOV}(c(\hat{\theta}) - q)^{-1} \cdot [c(\hat{\theta}) - q] \tag{32}$$

where $\text{ACOV}(c(\hat{\theta}) - q)$ denotes the asymptotic covariance of the estimated parameters given the restrictions. Under the H_0 , W is asymptotically χ^2 -distributed with degrees of freedom equal to the number of restrictions. Stated otherwise, the number of *df* equals the number of equations in $c(\hat{\theta}) - q = \mathbf{0}$.

In Equation (32), the asymptotic covariance of the estimated parameters given the restrictions $\text{ACOV}(c(\hat{\theta}) - q)$ can be obtained from the asymptotic covariance matrix of the estimated parameters $\text{ACOV}(\hat{\theta})$ by means of the delta method as

$$\text{ACOV}(c(\hat{\theta}) - q) = J[c(\hat{\theta})] \cdot \text{ACOV}(\hat{\theta}) \cdot J[c(\hat{\theta})]^\top \tag{33}$$

with $J[c(\hat{\theta})]$ being the Jacobian of the restriction function $c(\theta)$.

A set of J linear restrictions can be written in form of a restriction matrix R . The number of rows of the restriction matrix R corresponds to the number of restrictions and the number of columns corresponds to the length of the parameter vector θ . In this case, the hypothesis (31) becomes

$$H_0 : R \cdot \theta = q. \tag{34}$$

Because $J[R \cdot \hat{\theta}] = R$, following from Equation (33), the asymptotic covariance matrix of the estimated parameters given by the linear restrictions is $R \cdot \text{ACOV}(\hat{\theta}) \cdot R^\top$ and the equation for the Wald statistic becomes

$$W = [R \cdot \hat{\theta} - q]^\top \cdot [R \cdot \text{ACOV}(\hat{\theta}) \cdot R^\top]^{-1} \cdot [R \cdot \hat{\theta} - q], \tag{35}$$

see Greene (2012, p. 528). The number of *df* corresponds to the number of restrictions, i.e., the number of rows of R . Moreover, if $q = \mathbf{0}$, then the equation for the Wald test given the restrictions R further simplifies to

$$W = [R \cdot \hat{\theta}]^\top \cdot [R \cdot \text{ACOV}(\hat{\theta}) \cdot R^\top]^{-1} \cdot [R \cdot \hat{\theta}]. \tag{36}$$

In the rest of this tutorial, we restrict our consideration only to cases in which $q = \mathbf{0}$.

Up to now, the considerations referred to the Wald test in general. In the following, we apply the Wald test for linear restriction to the comparison of standardized regression coefficients. To test the hypothesis about standardized regression coefficients, we need the restriction matrix R , which is set up by the researcher, the vector of standardized parameters $\hat{\theta}^S$ and the asymptotic covariance matrix of the estimated standardized parameters $\text{ACOV}(\hat{\theta}^S)$. The asymptotic covariance matrix of the standardized estimated parameters $\text{ACOV}(\hat{\theta}^S)$ can be obtained from the asymptotic covariance matrix of the estimated parameters $\text{ACOV}(\hat{\theta})$ by using the delta method (Muthén, 2007; see also Greene, 2012, pp. 1083-1084). For instance, in *lavaan* there are functions that provide easy access to the required $\hat{\theta}^S$ and $\text{ACOV}(\hat{\theta}^S)$ so that no calculations by the user are needed.



In the following, we adopt Equations (38) and (39) for the comparison of standardized regression coefficients. For the sake of convenience, we define

$$M := [R \cdot \text{ACOV}(\hat{\theta}^S) \cdot R^T], \tag{37}$$

and the expression for Wald test for standardized parameters becomes

$$W = [R \cdot \hat{\theta}^S - q]^T \cdot M^{-1} \cdot [R \cdot \hat{\theta}^S - q], \tag{38}$$

or, if $q = 0$ respectively,

$$W = [R \cdot \hat{\theta}^S]^T \cdot M^{-1} \cdot [R \cdot \hat{\theta}^S]. \tag{39}$$

B Covariance matrices and variable description

B.1 Covariance matrices for the first and second example

Variable description (OECD, 2009):

η_1 Parental occupational status: International socio-economic index of occupational status (HISEI)

η_2 Parental education level: Education level of the parents (ISCED)

η_3 Home possessions: Summary index of all household items including the number of books at home

η_4 Educational resources: Index of items necessary for learning (desk, quiet place, computer, software, calculator, books, dictionary)

η_5 Reading ability: Performance of students in a reading task

	η_1	η_2	η_3	η_4	η_5
η_1	226.2577				
η_2	29.9232	9.0692			
η_3	4.8812	1.0312	0.8371		
η_4	1.9878	1.0043	0.2993	0.7084	
η_5	271.1429	49.5848	20.0337	15.7012	6088.8281

The following covariance matrix contains the rescaled variable \tilde{V}_3 :

	η_1	η_2	$\tilde{\eta}_3$	η_4	η_5
η_1	226.2577				
η_2	29.9232	9.0692			
$\tilde{\eta}_3$	48.8120	10.3120	83.710		
η_4	1.9878	1.0043	2.993	0.7084	
η_5	271.1429	49.5848	200.337	15.7012	6088.8281

B.2 Covariance matrix for the third example

Variable description (Schoon & Parsons, 2002, pp. 269-271):

y_1 Examination score: Score of the educational achievement at age 16

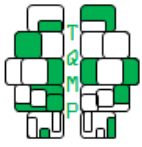
y_2 Highest qualification: Score in the Certificate of Secondary Education at age 16

y_3 Job aspiration: Open answer to the question “What do you expect to be your first full-time-job?” answered at age 16 and coded according to a coding scheme

y_4 Educational aspiration: Score indicating whether participants want to continue full-time education and obtain a degree

y_5 Goldthorpe: Erikson–Goldthorpe Scale, measuring social position based on dimensions of work-setting (being an employer or employee, performing manual on nonmanual work, the type of relationships between employees and employers)

y_6 RGSC: Standard Occupational Classification of Registrar General’s social class (RGSC)



	y_1	y_2	y_3	y_4	y_5	y_6
y_1	263.413					
y_2	5.996	0.436				
y_3	2.312	0.067	0.212			
y_4	7.591	0.213	0.164	0.722		
y_5	78.740	2.271	1.295	4.125	244.923	
y_6	6.622	0.190	0.110	0.347	14.648	1.440

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