Interactive Parallel Models:
No Virginia, Violation of Miller's Race Inequality does not Imply Coactivation and
Yes Virginia, Context Invariance is Testable

James T. Townsend [a] ✉ ⓘ , Yanjun Liu [a] ⓘ , Ru Zhang [b] ⓘ & Michael J. Wenger [c] ⓘ

[a]Indiana University
[b]Boys Town National Research Hospital
[c]University of Oklahoma

Abstract ∎ One vein of our research on psychological systems has focused on parallel processing models in disjunctive (OR) and conjunctive (AND) stopping-rule designs. One branch of that research has emphasized that a common strategy of inference in the OR situations is logically flawed. That strategy equates a violation of the popular Miller race bound with a coactive parallel system. Pointedly, Townsend & Nozawa (1997) revealed that even processing systems associated with extreme limited capacity are capable of violating that bound. With regard to the present investigation, previous theoretical work has proven that interactive parallel models with separate decision criteria on each channel can readily evoke capacity sufficiently super to violate that bound (e. g., Colonius & Townsend, 1997; Townsend & Nozawa, 1995; Townsend & Wenger, 2004). In addition, we have supplemented the usual OR task with an AND task to seek greater testability of architectural, decisional, and capacity mechanisms (e. g., Eidels et al., 2011; Eidels et al., 2015). The present study presents a broad meta-theoretical structure within which the past and new theoretical results are embedded. We further exploit the broad class of stochastic linear systems and discover that interesting classical results from Colonius (1990) can be given an elegant process interpretation within that class. In addition, we learn that conjoining OR with AND data affords an experimental test of the crucial assumption of context invariance, long thought to be untestable.

Introduction

Imagine a driver approaching a busy intersection. She sees both a red light and a pedestrian entering the crosswalk. Either one of these should cause the driver to stop. In fact, she should stop based on the first one of these that she sees. This is a task with two consistent signals and a *disjunctive* or *OR decision rule*. Her response times (*RTs*) should reflect the minimum time to process the two signals. Other logical rules, the most obvious being a logical *conjunction* (an *AND decision rule)*, are possible. For example, the driver should wait until both the light changes to green and the pedestrian is out of the intersection before she proceeds. If the driver is forced to process one signal at a time, *serial processing* ,then unless there is a terrific speed-up when more than one signal appears, performance will be much slower with the greater number of signals, that is, with an increased workload. Thus, the typical serial model will be what is known as limited capacity but even when both signals can be processed simultaneously (*parallel processing*), the system can be quite limited in its performance (Townsend, 1972).

Figure 1 ■ (a) A basic serial model where the processing of internal channels occurs in serial order. (b) A basic parallel model where the processing of internal channels occurs simultaneously (c) A basic coactive model where the information from each internal channel sums together before the decision stage.

(b)

(a)

(c)
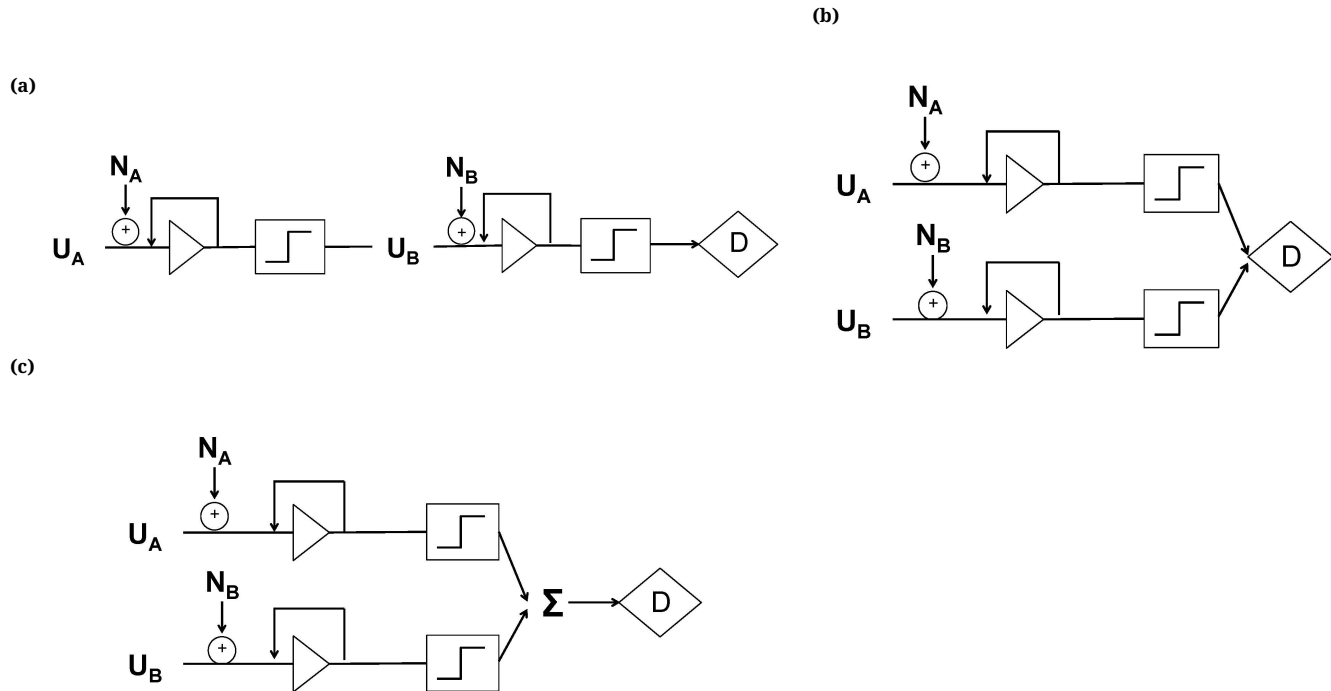
The perception-action example just presented represents a focus of intense interest in both pure and applied psychology over many years. Yet, parallel processing is purported to be ubiquitous in the brain and almost any psychological task involving more than one sub-process raises questions of parallel vs. other architectures, including serially arranged systems. For an up to date account of our general approach to the developement of theory-driven methodologies, see Algom, Eidels, Hawkins, Jefferson, and Townsend (2015) and manifold applications in Algom, Fitousi, and Eidels (2017). Thus, the fundamental properties of elementary psychological systems, including parallel architectures, continue to be of high importance. In addition to the question of pure knowledge concerning basic psychological processing systems, there are several fields in perception, cognition and reaction of great historical and contemporary interest in psychology. One that reaches back into the nineteenth century but remains of high importance in modern psychology pertains to Gestalt Psychology and its modern referents such as initiated by Wendell R. Garner and others (e.g., see the recent in-depth surveys and reviews in Algom & Fitousi, 2016; Fitousi & Algom, 2018; Colonius & Diederich, 2018).

Figure 1 shows a *serial* (panel a), *parallel* (panel b), and a *coactive model* (panel c) where activation of paralled

channels are summed before a decision. Here, D the decision could refer to any logical decision gate (e.g., OR or AND) but in the case of the coactive system, D would generally simply signify that the activation achieves a decisional criterion. Subsequently, our language will precisely indicate the type of decision envisioned.

Now, ordinary parallel models, which possess decision bounds (i.e., criteria) for each channel will be contrasted below, as in the literature, with coactive models, but only the parallel models are under quantitative scrutiny here. In fact, the *capacity* of parallel systems—their response to changes in workload—could be affected not only by workload but also by whether the channels *interact* or not and, if they do, whether the interactions are *positive (facilitatory)* or *negative (inhibitory)*. We focus our analyses on parallel systems with channels that interact.

There are now viable parallel models, defined mathematically, that have been successfully applied to many empirical phenomena. These include but are not limited to the *theory of visual attention* (TVA, e.g., Bundesen, 1990), the *linear ballistic accumulator* (LBA, e.g., Brown & Heathcote, 2008), and the *predictive interactive multiple memory systems framework* (Henson & Gagnepain, 2010).

Our present concern though, is with fundamental properties of broad classes of parallel systems rather than spe-

cific models, in the spirit of theoretical work by Dzha-farov (1993), Smith and Van Zandt (2000), and Townsend, Wenger, and Houpt (2018). In fact, to elaborate a bit further, our focus will be on the analysis of *interactive parallel channels* and comparison with *stochastically independent parallel channels* which do not vary in efficiency as workload is increased. In that sense, our quantitative investigation continues the tradition of Colonius (1990), Colonius and Vorberg (1994), Colonius and Townsend (1997), Townsend and Wenger (2004, 2014), and Eidels, Houpt, Altieri, Pei, and Townsend (2011).

The overall goals of the present investigation then, are as follows: We start by providing background on the psychological interest in these kinds of systems. A major portion of that history revolves around the study of *redundant signals* in OR tasks (see background in Townsend & Nozawa, 1995). We put forth a strong objection to the most popular and long-standing mode of connecting theory and data in such studies.

We next formulate a general theory of models of independent vs. interactive parallel systems for OR and AND decision situations. Within that theory, we recall seminal results of Colonius and colleagues (Colonius, 1990; Colonius & Vorberg, 1994). This type of model is more descriptive because it merely assumes a joint processing time distribution on the parallel channels and then imposes a stopping rule.

Next, we consider models that assume *activation-state spaces* in which channel activations accrue up to decision thresholds. This tactic brings us square into the topic of the present special issue of TQMP: *accumulator or sequential sampling models*. Of course, in our case, each parallel channel becomes a sequential sampling system. In the near past we have referred to this entire class as *accrual halting models* (Townsend, Houpt, & Silbert, 2012).

We thereby introduce a specific type of these accrual halting models which are, in fact, *stochastic linear systems*. We call upon members of this class in order to characterize and demonstrate various principles of parallel systems. We then report not only systems that, as in Townsend and Wenger (2004), violate the predictions of Colonius and colleagues, but important and up to now, novel stochastic linear systems, including some that perfectly capture the properties described by Colonius (1990). We further suggest that this set of models serves valuable purposes of taxonomy in which to explore theoretical issues and interpret data.

A critical assumption in the field of race models, reviewed and analyzed in detail below, that has played a dominant role in multi-sensory perception, is that of *context invariance*. For approximately 40 years this assumption has been thought to be empirically untestable (see

e.g., Lombardi, D'Alessandro, & Colonius, 2019; Luce, 1986; Ashby & Townsend, 1986). A contribution of the present investigation is that we demonstrate that if the experimenter utilizes both an OR (i.e., disjunctive stopping rule) as well as an AND (i.e., conjunctive stopping rule), then certain types of outcomes concerning workload capacity predictions, can falsify context invariance.
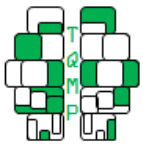
Our concluding discussion points out some recent contributions to the literature that appear compatible with our own message (e. g., Brown & Heathcote, 2008; Henson & Gagnepain, 2010). We then reiterate our primary findings and conclusions and envision promising future research possibilities. Finally, for reference, Table 1 summarizes some of the most critical equations and definitions in the material that follows.

A Brief History and Background

The performance of parallel models is intriguing from a psychological standpoint, and has foundational importance for understanding human information processing. This is because, for example, more signals might be perceived faster than few in an OR situation simply because, statistically, the probability of the fastest time out of n, being quicker than say time t, will be greater than that for $k < n$ (Colonius & Townsend, 1997; Gumbel, 1958; Logan, 1988; Raab, 1962; Townsend & Colonius, 2005). This will certainly be true for parallel channels, each of whose speed does not change as the number of channels changes. It will remain true, but decreasingly so, for parallel models whose channels slow down as the number of engaged channels increments (due to increased workload) up to some limit of slowing down of these channels. At such a point, the overall minimum processing time will degrade to be worse than that of an original single channel.

On the other hand, performance might speed up or slow down when multiple channels are operating on, simply depending on whether the signals are congruent or incongruent (e. g., Fan, McCandliss, Sommer, Raz, & Posner, 2002; Wenger & Townsend, 2006; Eidels, Townsend, & Algom, 2010; Shalev & Algom, 2000). The general topic of how performance varies with number of pertinent signals has been a popular one for at least half a century now (e. g., Sperling, 1960; Raab, 1961; Estes & Taylor, 1964; Egeth, 1966; Sternberg, 1966; Bernstein, Rose, & Ashe, 1970; Nickerson, 1973), including work in areas as disparate as applied decision-making (e. g., Patterson et al., 2013) and multimodal perception (e. g., Calvert, Spence, Stein, et al., 2004; Otto & Mamassian, 2012). Modern literature often refers to this topic as that of *workload capacity*.

Considerable experimental and methodological effort (i. e., Colonius, 2016; Lombardi et al., 2019) has followed papers by J. Miller and colleagues (1978, 1982, 2016) in

Table 1 ∎ Summary of critical equations and definitions.

| Name	Equation
Grice bound	$P_{AB}(T_A \leq t \text{ or } T_B \leq t) \geq \text{MAX}\left[P_A\left(T_A \leq t\right), P_B\left(T_B \leq t\right)\right]$
Miller's bound	$P_{AB}(T_A \leq t \text{ or } T_B \leq t) \leq \text{MIN}\left[P_A\left(T_A \leq t\right) + P_B\left(T_B \leq t\right), 1\right]$
C-V lower bound	$P_{AB}(T_A \leq t \text{ and } T_B \leq t) \geq \text{MAX}\left[\left(P_A\left(T_A \leq t\right) + P_B\left(T_B \leq t\right) - 1\right), 0\right]$
C-V upper bound	$P_{AB}(T_A \leq t \text{ and } T_B \leq t) \leq \text{MIN}\left[P_A\left(T_A \leq t\right), P_B\left(T_B \leq t\right)\right]$
Capacity for OR	$C_{OR}(t) = \frac{H_{AB}(t)}{H_A(t) + H_B(t)} = \frac{ln(S_{AB}(t))}{ln(S_A(t)) + ln(S_B(t))}$, where $H(t) = \int_0^t \frac{f(t')}{S(t')} dt'$, $S(t) = 1 - F(t)$
Capacity for AND	$C_{AND}(t) = \frac{K_A(t) + K_B(t)}{K_{AB}(t)} = \frac{ln(F_A(t)) + ln(F_B(t))}{ln(F_{AB}(t))}$, where $K(t) = \int_0^t \frac{f(t')}{F(t')} dt'$

which he advanced an inequality on RTs along with some operational definitions connecting it to possible underlying psychological themes. That inequality has come to be called *the Miller race bound*. A lower bound on performance was suggested and utilized by Grice and colleagues (e. g., Grice, Canham, & Boroughs, 1984) and this inequality (*the Grice bound*) will enter into our treatment as well. Soon, the equally valuable conjunctive (AND case) stopping rule was introduced along with upper and lower bounds on RT performance, although it has received far less experimental attention (e.g., Colonius & Vorberg, 1994). More formal theoretical treatments appeared in the 1990s, for example, Colonius (1990), Schwarz and Ischebeck (1994), Diederich and Colonius (1991), Miller (1991), Colonius and Townsend (1997), Townsend and Nozawa (1995) and Townsend and Wenger (2004).

Almost all the emphasis in empirical work has been whether or not the race inequality was violated. If a violation occurred, the inference, according to the operational definition (Miller, 1982), was that the system was coactive (i.e., Schröger & Widmann, 1998; Besle, Fort, Delpuech, & Giard, 2004; Leone & McCourt, 2015), a particular variant of parallel processing as we noted. Even before initiating our present more technical analyses, we pause to note that this had to be falacious or at least exceedingly suspect due to our earlier demonstrations that quite pedestrian systems, even serial processing systems, could cause violations of that bound (Townsend & Nozawa, 1997). In any event, coactivation began to receive a more theoretical interpretation by the early nineties (e. g., Diederich & Colonius, 1991; Schwarz & Ischebeck, 1994; Townsend & Nozawa, 1995; Colonius & Townsend, 1997). The consensus in all the theoretical accounts was that activation from independent parallel channels would feed into a fi-

nal conduit in which the summed activation was compared against a single decision criterion. This type of model is exhibited in Figure 1c. The quantitative analyses verified not only that coactive models could support performance superior to the standard parallel model with independent channels and context invariance, but had to violate the race inequality (see Townsend & Nozawa, 1995).

Thus, almost all experimental papers have simply concluded ordinary race processing if the inequality was satisfied or coactivation if it was not. Lately, some studies have shifted from a conclusion of coactivation on discovery of violation of the race inequality and have employed more neutral but sometimes rather vague language. For instance, some investigators have used terms such as *binding*, *integration*, *multi-modal advantage*, etc. (e. g., Raij, Uutela, & Hari, 2000; Harrar, Harris, & Spence, 2017; Noel, Modi, Wallace, & Van der Stoep, 2018).

Our inquiry will not focus on coactivation but rather with ordinary parallel processing where each channel has its own decisional criterion. Figure 1b shows a basic parallel model. Our new theoretical developments expand our earlier Townsend and Wenger (2004) work in the following ways: (a) the class of race models is far larger than those that assume context invariance and that are typically referred to in the literature; and (b) a considerable sub-class of race models can produce not only super capacity but also violations of the race inequality. Moreover, our general scheme encourages the perspective of a continuum of capacity effects and due attention to the fact that a substantial number of experiments find performance lying in the range of unlimited capacity (defined rigorously below) to mild limitations in capacity, far from the upper race bound as well as the lower Grice bound (also defined below). In fact, we have demonstrated that even the famous Stroop
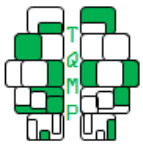
stimuli can elicit performance in this very moderate range (Eidels et al., 2010). Proceeding in that direction, it is necessary to summarize our informal but crucial definition. A parallel race model assumes (a) parallel processing channels possessing joint distributions on the individual processing times per channel; (b) in OR situations, the first channel to finish determines the RT (i.e., a *minimum time stopping rule*), while in AND situations, the overall processing time demands that both channels have completed processing (i.e., a *maximum time stopping rule*); and (c) there may or may not be positive or negative interactions across channels, along with noise.

So-called standard parallel models assume not only stochastically independent channels but also invariance of the marginal processing time distributions of each channel (Algom et al., 2015; Zhang, Liu, & Townsend, 2018, 2019). Colonius (e. g., Colonius, 1990; Ashby & Townsend, 1986; Luce, 1986) referred to this as the assumption of *context independence*. We have since suggested the term *context invariance* to distinguish this concept from that of true stochastic independence. Context invariance might fail even in models with stochastically independent channels as the workload changes, for instance, through limited resources. However, the main alternative is alterations due to channel interactions and those are the focus of the present inquiry.

In the early theoretical work by Colonius (1990) it was recognized that a different reason for violations of the race inequality could be the failure of context invariance (see page 269 in Colonius, 1990; Colonius & Townsend, 1997). Thus, when the workload (i.e., the number of signals) is increased, the average speed of individual channels may or may not be affected. The more technical term for average speed is *marginal processing time distribution* and we will use that terminology. Nonetheless, outside of the discussion in Townsend and Wenger (2004), there has not been much attention to this rather important aspect. We demonstrated there that many reasonable parallel models which allowed cross-channel interactions fail to preserve the channels' marginal distributions. This possibility can deliver experimental conclusions that are quite distinct from those permitted under the assumption of context invariance. However, many strategic issues have remained untouched and, in one important case, incorrectly understood. In particular, and importantly, we show that combining experimental results from both OR and AND conditions, presumably from the same participants, can sometimes falsify the critical assumption of context invariance.

In our earlier theoretical explorations (Townsend & Wenger, 2004), it was claimed that realistic stochastic systems such as those employed here were incapable of re-

alizing the Colonius (1990) regularities. We were wrong in this assertion. Our present set of more extensive and penetrating analyses not only finds a type of stochastic system that can perfectly realize the Colonius (1990) predictions but also demonstrates that other arrangements can approximate these.

Establishing the Standard Parallel Model and the Fundamental Inequalities for OR and AND Processing Times

Performance expressed in terms of RTs can be written in terms of various statistics, including the cumulative distribution functions as in Miller (1982) and many papers that have ensued. An alternative that has been found to be useful for many purposes has been what has been called the *workload capacity function*, $C_i(t)$, where the "i" is used to designate the kind of decisional stopping rule in force (Townsend & Nozawa, 1995; Townsend & Wenger, 2004; Townsend & Colonius, 2005), that is, i = OR or AND. However, because much of our discussion revolves around the marginal processing time distributions vs. the joint distribution on processing times, the first technical sections will avoid the $C_i(t)$ machinery.

In the present section, we summarize the critical regularities to establish a reference point for the developments that follow. The simplest case for OR processing concerns two inputs. These inputs can be presented either alone or together, and when presented together are processed using a disjunctive decisional strategy, represented by the Boolean OR operator. Here we assume two channels, A and B, with (unobservable) processing times T_A and T_B.[1] From elementary probability theory, we know that when both signals are present

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ or } T_B \leq t) = \\
P_{AB}(T_A \leq t) \\
+ P_{AB}(T_B \leq t) \\
- P_{AB}(T_A \leq t \text{ and } T_B \leq t).
\end{aligned}
\tag{1}
$$

It has long been known that if the channel times are stochastically independent, then

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ or } T_B \leq t) = \\
P_{AB}(T_A \leq t) \\
+ P_{AB}(T_B \leq t) \\
- P_{AB}(T_A \leq t) P_{AB}(T_B \leq t).
\end{aligned}
\tag{2}
$$

Now, if the marginal processing times are unchanged when going from single to double signal trials (i.e., if context invariance holds), and if the channels are independent, then

---

[1] These are to be distinguished from observable RTs.
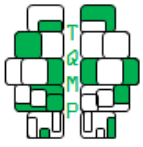
we get

$$P_{AB}(T_A \leq t \text{ or } T_B \leq t) =$$
$$P_A(T_A \leq t)$$
$$+P_B(T_B \leq t)$$
$$-P_A(T_A \leq t)P_B(T_B \leq t). \tag{3}$$

We refer to this combination of independence together with invariant marginals as being parallel with stochastically independent channels, with unlimited capacity, and we refer by shorthand, to this conception as the *standard parallel model*.

Next, note that if the left-hand side of Equation 1 is larger than or equal to either $P_{AB}(T_B \leq t)$ or $P_{AB}(T_A \leq t)$ alone, and therefore the biggest of the two then:

$$\text{MAX}[P_{AB}(T_A \leq t),\ P_{AB}(T_B \leq t)] \leq$$
$$P_{AB}(T_A \leq t \text{ or } T_B \leq t) \tag{4}$$

Satisfaction of the condition that $P_{AB}(T_A \leq t \text{ or } T_B \leq t)$ is greater than either of the single signal processing times becomes the well-known *Grice inequality* (e. g., Grice et al., 1984):

$$P_{AB}(T_A \leq t \text{ or } T_B \leq t) \geq$$
$$\text{MAX}[P_A(T_A \leq t), P_B(T_B \leq t)] \tag{5}$$

If overall processing time is less than predicted by Equation 3 then the system is designated as exhibiting limited capacity. And, if the Grice inequality, Equation 4, is violated, performance is considered to be extremely limited capacity (Townsend & Wenger, 2004; Townsend & Nozawa, 1995; Townsend & Eidels, 2011).

In contrast to limited capacity, what level of performance would be superior to that of the standard parallel model (Equation 3)? As observed earlier, Miller (e.g., 1978, 1982) brought to bear Boole's inequality to address this question:

$$P_{AB}(T_A \leq t \text{ or } T_B \leq t) \leq$$
$$P_A(T_A \leq t) + P_B(T_B \leq t) \tag{6}$$

If the inequality in Equation 3 is violated, then performance must be superior to what would be expected from the standard parallel model. Our approach views performance that is faster than that associated with Equation 3 as super capacity, and if the *Miller race inequality*, Equation 6, is violated, as extreme super capacity (Townsend & Nozawa, 1995; Townsend & Wenger, 2004; Townsend & Eidels, 2011). So far then, the logic is that if performance for parallel race models is neither extreme super capacity nor extremely limited capacity, the data should lie between the

Miller and Grice's bounds:

$$\text{MAX}[P_A(T_A \leq t),\ P_B(T_B \leq t)] \leq$$
$$P_{AB}(T_A \leq t \text{ or } T_B \leq t) \leq \tag{7}$$
$$\text{MIN}[P_A(T_A \leq t) + P_B(T_B \leq t), 1]$$

Next, consider a pair of independent parallel channels that, when processing two inputs, determines the output on the basis of a conjunctive decisional strategy, represented by the Boolean AND operator. It is then required that we develop a set of results corresponding to those in Equations 7 for the AND case. To cut to the chase, and referring the interested reader to full discussions in earlier papers (e. g., Townsend & Wenger, 2004; Algom et al., 2015), the bounds corresponding to Equation 7 for the AND decision rule are:

$$\text{MAX}[(P_A(T_A \leq t) + P_B(T_B \leq t) - 1), 0] \leq$$
$$P_{AB}(T_A \leq t \text{ and } T_B \leq t) \leq \tag{8}$$
$$\text{MIN}[P_A(T_A \leq t), P_B(T_B \leq t)]$$

So if performance lies between the standard parallel model (Equation 3) and the upper bound (Equation 8), it is defined as moderately super capacity, and if greater than that, extreme super capacity. Likewise, if processing times are stochastically slower than that prescribed by Equation 3 but greater than the lower bound of Equation 8 the model is said to be moderately limited capacity and, if below that, extremely limited capacity.

How Channel Dependencies Affect Overall Speed Relative to the Fundamental Inequalities

The first general examination of the effects of stochastic dependencies on parallel processing was provided by Colonius (Colonius, 1990; Colonius & Vorberg, 1994). The pellucid logic of this approach was that the extent to which performance is faster or slower than what an unlimited capacity, independent, parallel model predicts is a function of the sign and magnitude of the stochastic dependence between the channel processing times. We begin with a summary of what is known with respect to self-terminating (OR) processing.

Positive dependence is naturally defined by the property that

$$P_{AB}(T_A \leq t \text{ and } T_B \leq t) > P_{AB}(T_A \leq t)P_{AB}(T_B \leq t)$$

and negative dependence is defined as the joint distribution of T_A and T_B being less than $P_{AB}(T_A \leq t)P_{AB}(T_B \leq t)$. The influence of the dependency's magnitude as well as its sign, is quite intuitive since this joint probability, $P_{AB}(T_A \leq t \text{ and } T_B \leq t)$ is subtracted from the sum of the marginals (Equation 3). But it can be observed that the
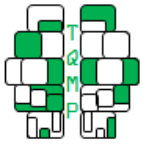
stochastic independence by itself offers no indication concerning the effect of moving from one signal to two. The product of terms in Equation 3 could be larger than the marginals when only one signal is present (producing limited capacity), or smaller (producing super capacity).

When compared with Equation 3, and when moving from single to the double signal trials, a positive correlation produces slower performance than an unlimited capacity race model, whereas a negative correlation elicits faster processing times. This can be seen by recalling the basic OR formulation:

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ or } T_B \leq t) = \\
P_{AB}(T_A \leq t) \\
+ P_{AB}(T_B \leq t) \\
- P_{AB}(T_A \leq t \text{ and } T_B \leq t)
\end{aligned}
\tag{9}
$$

When the marginals are constant (i.e., with marginal invariance) when moving from single-signal trials to double-signal trials,

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ or } T_B \leq t) = \\
P_A(T_A \leq t) \\
+ P_B(T_B \leq t) \\
- P_{AB}(T_A \leq t \text{ and } T_B \leq t)
\end{aligned}
\tag{10}
$$

we see that a positive dependence is defined by reduced speed in a probabilistic sense, and increased speed when there is a negative dependence, both relative to the standard parallel case.

Colonius's (1990) insight was to now bring to bear results established by Höffding (1940), Fréchet (1951) and others. Colonius determined that, assuming invariance of the marginal probabilities, the greatest speed-up in unlimited capacity race models that can occur with a negative dependence actually attains the Miller race bound, that is,

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ or } T_B \leq t) = \\
P_A(T_A \leq t) \\
+ P_B(T_B \leq t)
\end{aligned}
\tag{11}
$$

in the region where the sum is not greater than 1. Similarly, the most extreme slowing down that can occur with a positive dependence is equivalent to the Grice bound. That is, importantly, this lower bound is reached when the dependence reaches its positive extreme:

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ or } T_B \leq t) = \\
\text{MAX}[P_A(T_A \leq t), P_B(T_B \leq t)]
\end{aligned}
\tag{12}
$$

Now, consider the AND stopping rule, which was examined by Colonius and Vorberg (1994). The situation is opposite to that of the OR case: with a strictly positive correlation, and with marginal invariance, it follows that $P_{AB}(T_A \leq t \text{ and } T_B \leq t) > P_A(T_A \leq t)P_B(T_B \leq t)$. Therefore, a positive correlation now facilitates performance rather than degrading it, as it did in the OR case, because the marginals do not directly enter these mathematical expressions. The converse holds for a negative dependence. A maximal positive correlation elicits

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ and } T_B \leq t) = \\
\text{MIN}[P_A(T_A \leq t), P_B(T_B \leq t)]
\end{aligned}
\tag{13}
$$

and when a negative dependency is maximized

$$
\begin{aligned}
P_{AB}(T_A \leq t \text{ and } T_B \leq t) = \\
P_A(T_A \leq t) \\
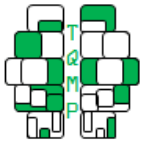+ P_B(T_B \leq t) \\
- 1,
\end{aligned}
\tag{14}
$$

as long as the right-hand sum is great than or equal to 0.

At this juncture, we have established that in the presence of context invariance, positive dependencies hurt OR performance and negative dependencies help it. Conversely, in AND processing, positive dependencies enhance performance whereas negative dependencies degrade performance. How can the discussion be deepened and expanded? The most natural expansion is to bring in the notion of states of processing, as many if, not most, models of RT in recent psychology are, indeed, like sequential sampling models,founded on the concept of state spaces and their stochastic behavior over time (e. g., Link & Heath, 1975; Ratcliff, 1978; Townsend & Ashby, 1983; Busemeyer & Townsend, 1993; Donkin, Brown, & Heathcote, 2011; Smith & Van Zandt, 2000).

Parallel Systems Based on Activation Accrual within Channels

The need to explore inter-channel dependencies motivated us to investigate very general parallel systems whose channels accrue activation until a decision threshold or criterion is reached. As observed earlier, we have termed such models accrual halting models (Townsend et al., 2012). This class contains a considerable number of models, including all sequential sampling models.

In this approach, each parallel channel is conceptualized as a stochastic process on accumulating information or activation, as illustrated in Figure 4a. In this representation U_j denotes the input value to each channel, and N_j denotes the white noise in each channel, where $j = A, B$. The activation coefficients a and b in each channel represent stabilizing self-feedback. These parallel channels may or may not interact (Townsend & Wenger, 2004; Townsend & Altieri, 2012). Activation or even noise associated with one channel may feed over, in a positive or negative fashion, to the other channel.

In the current work, for the case where the marginal probabilities can vary, we introduce dependencies in the system by allowing cross-channel talk between channel A and B. Specifically, α and β are cross channel activation coefficients. If $\alpha > 0$, then the output of channel B is positively dependent on channel A. If $\alpha < 0$, then the output of channel B is negatively dependent on channel A, and vice versa for β. The reader may observe the important facet that for the invariant marginals case, the dependency in the system is introduced by a common noise to both channel A and B.

A critical aspect of accrual halting systems is that a positive dependence in the activation in two channels, A and B, implies a positive dependence in the completion times of the two channels. This is straightforward to understand as the stopping times are completely tied up with the time point when activation reaches a threshold c_j:

$$P(T_j \leq t) = P\{[\text{MIN T such that } X_j(T) \geq c_j] \leq t\}, \quad j = A, B \tag{15}$$

The intuition is that in, say a parallel system with positively correlated activated channels, the probability that $X_A > c_A$ given that $X_B > c_B$ at time t, is the same as stating the probability that $T_A \leq t$ given that $T_B \leq t$ and these are greater than the marginal probability that $T_A \leq t$.

As noted earlier, Townsend and Wenger (2004), Wenger and Townsend (2006) studied a range of models based on noisy linear systems, with special emphasis on interacting vs. independent channels. With these models, it was straightforward to use positive dependencies to violate the Miller bound, although this outcome was avoided with very weak interactions. If the channels were instead configured to inhibit one another, then Grice's inequality was violated with moderate to high levels of negative interaction. If one takes these channel dependencies to represent correlations, then these results are inconsistent with the predictions associated with the assumption of context invariance.

Now, for some time, all our applications tacitly accepted the premise that correlated systems facilitate speed if the correlation is positive and cause slower processing times if the correlation is negative (Townsend & Wenger, 2004; Wenger & Townsend, 2006; Eidels et al., 2011). However, as will be shown below, we were wrong: There do exist simple stochastic linear systems that embody the Colonius (1990) logic rather than our own. First, we need to recall the fundamentals of the workload capacity functions. These will make our statements of comparison and inferences much easier to understand.

Brief Tutorial on the Workload Capacity Functions

The capacity coefficient (Townsend & Nozawa, 1995; Townsend & Eidels, 2011) is a distribution-free response-time measure used to index the processing efficiency change as a function of alterations in the number of internal channels of a person's cognitive processing. Townsend and Eidels (2011) developed a unified set of indexes of performance for OR and AND designs, that include the upper and lower bounds discussed earlier. Blaha and Houpt (2016) presented statistic bounds comparable for situations where there exists one target accompanied by a single distractor, the well known case of single-target, self-terminating stopping rule. However, as noted earlier, the emphasis here will be on the OR and AND designs.

In order to specify the capacity measure for the pertinent stopping rule, we must append a subscript to indicate whether, an OR or AND rule is in effect; thus, $C_i(t)$ where $i = $ OR or AND. Either capacity coefficient $C_i(t)$, as shown in Equation 16 and 17, can be regarded as a ratio comparing the performance of the system in a double-channel condition to the summation of performances from the single channels.

$$\begin{aligned} C_{OR}(t) &= \frac{H_{AB}(t)}{H_A(t) + H_B(t)} \\ &= \frac{\ln(S_{AB}(t))}{\ln(S_A(t)) + \ln(S_B(t))}, \end{aligned} \tag{16}$$

where

$$H_i(t) = \int_0^t \frac{f_i(t')}{S_i(t')} dt'$$

and

$$\begin{aligned} C_{AND}(t) &= \frac{K_A(t) + K_B(t)}{K_{AB}(t)} \\ &= \frac{\ln(F_A(t)) + \ln(F_B(t))}{\ln(F_{AB}(t))}, \end{aligned} \tag{17}$$
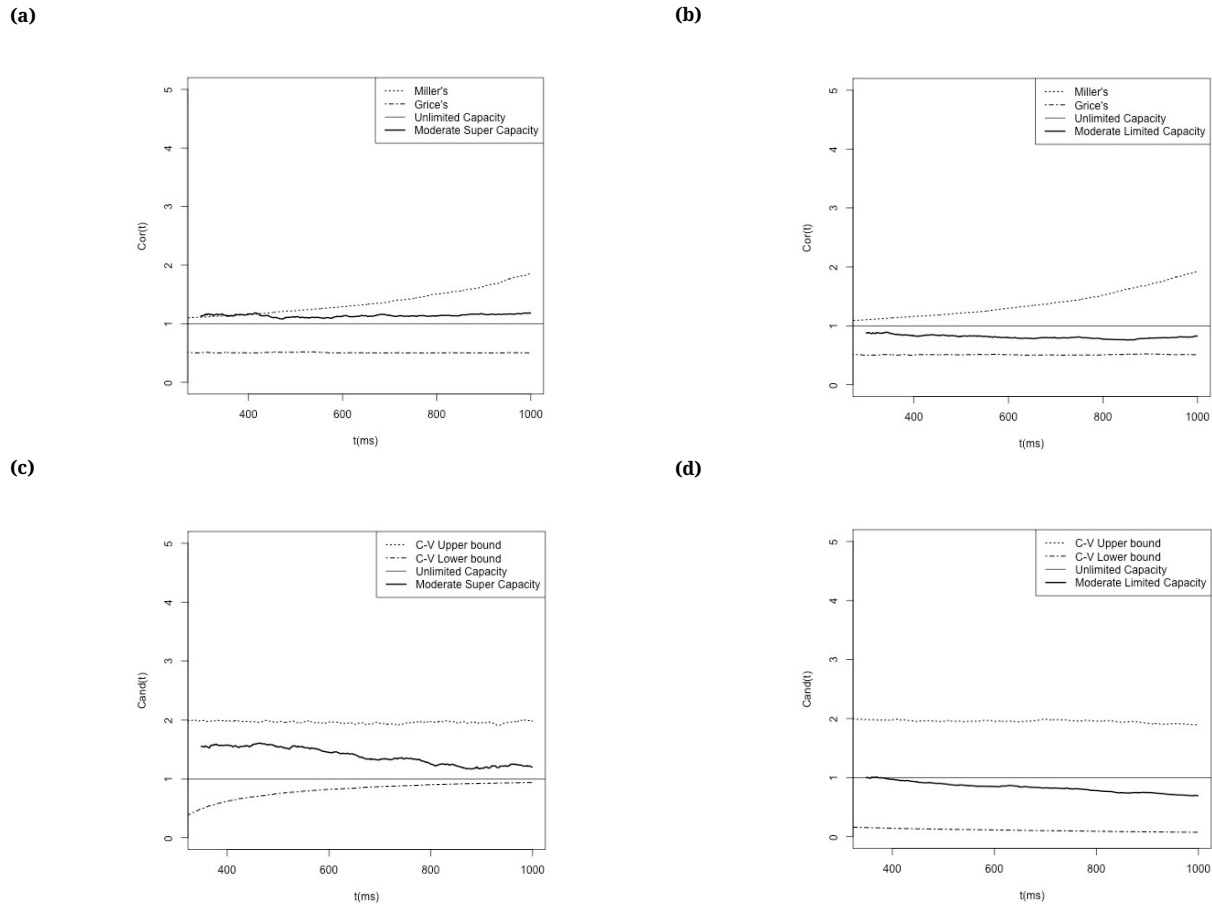
where

$$K_i(t) = \int_0^t \frac{f_i(t')}{F_i(t')} dt'$$

For both versions (OR, AND) of the capacity coefficient, if $C_i(t) = 1$, then the system is defined as possessing unlimited capacity because that is the level associated with the standard parallel model as described above. If $C_i(t) > 1$, the system is said to possess super capacity at t. And, this level of $C_i(t)$ indicates that the increasing number of internal channels actually improves the total processing efficiency of the system. Super capacity is often associated with holistic processing or a coactivation system (e. g., Wenger, Schuster, & Townsend, 2002; Townsend & Wenger, 2014). If $C_i(t) < 1$, the system is deemed to be of limited capacity, implying that the increase in number of internal channels impairs the overall performance of the system.

We implement a set of simulations of our linear systems to illustrate the regularities. The simulation methods

Figure 2 ∎ (a) A special case for OR where for all $t > 0$, the system possesses moderately super capacity. (b) A special case for OR where for all $t > 0$, the system possesses moderately limited capacity.(c) A special case for AND where for all $t > 0$, the system possesses moderately super capacity. (d) A special case for AND where for all $t > 0$, the system possesses moderately limited capacity.

(a)

(b)

(c)

(d)

are described in the Appendix. Figure 2a and b illustrate special cases for OR where for all $t > 0$, the system is, respectively, unlimited, or moderately super, or moderately limited, in terms of $C_{OR}(t)$. Figure 2c and d depict cases for AND with moderate super and limited capacity, respectively. Recall that extreme super capacity is inferred if an upper bound is violated and extreme limited capacity is inferred if a lower bound is violated. Then, Figure 3a and b show instances of OR processing with extreme super or extreme limited capacity where as Figure 3c and d show instances of AND processing with extreme super or extreme limited capacity, respectively.

OR and AND Parallel Accrual Systems Permitting Variable Marginals

The details of channel interactions considered here differ from those of Townsend and Wenger (2004). Specifically,

the interaction here takes place *after* integration whereas in the former study the cross-talk was via recurrent feedback from the output back to be added to the input signal plus noise. In spite of this difference, we will see that the qualitative effects are quite similar.

We will now establish, through simulations, that our present class of stochastic linear systems enfolds all the basic types of capacity effects that might be expected from parallel channels that are either independent, positively dependent, or negatively dependent. These include but go beyond the earlier findings of Townsend and Wenger (2004). Further sections will also reveal that they can capture the precise properties of the context invariant systems of Colonius and colleagues (Colonius, 1990; Colonius & Vorberg, 1994). We pause to stress that our class of models is not in competition with Colonius's findings. The former, in the whole, is unfalsifiable where the latter is fal-

Figure 3 ■ (a) A special case for OR where for all $t > 0$, the system possesses extreme super capacity. (b) A special case for OR where for all $t > 0$, the system possesses extremely limited capacity. (c) A special case for AND where for all $t > 0$, the system possesses extreme super capacity. (d) A special case for AND where for all $t > 0$, the system possesses extremely limited capacity.

(a)

(b)

(c)

(d)
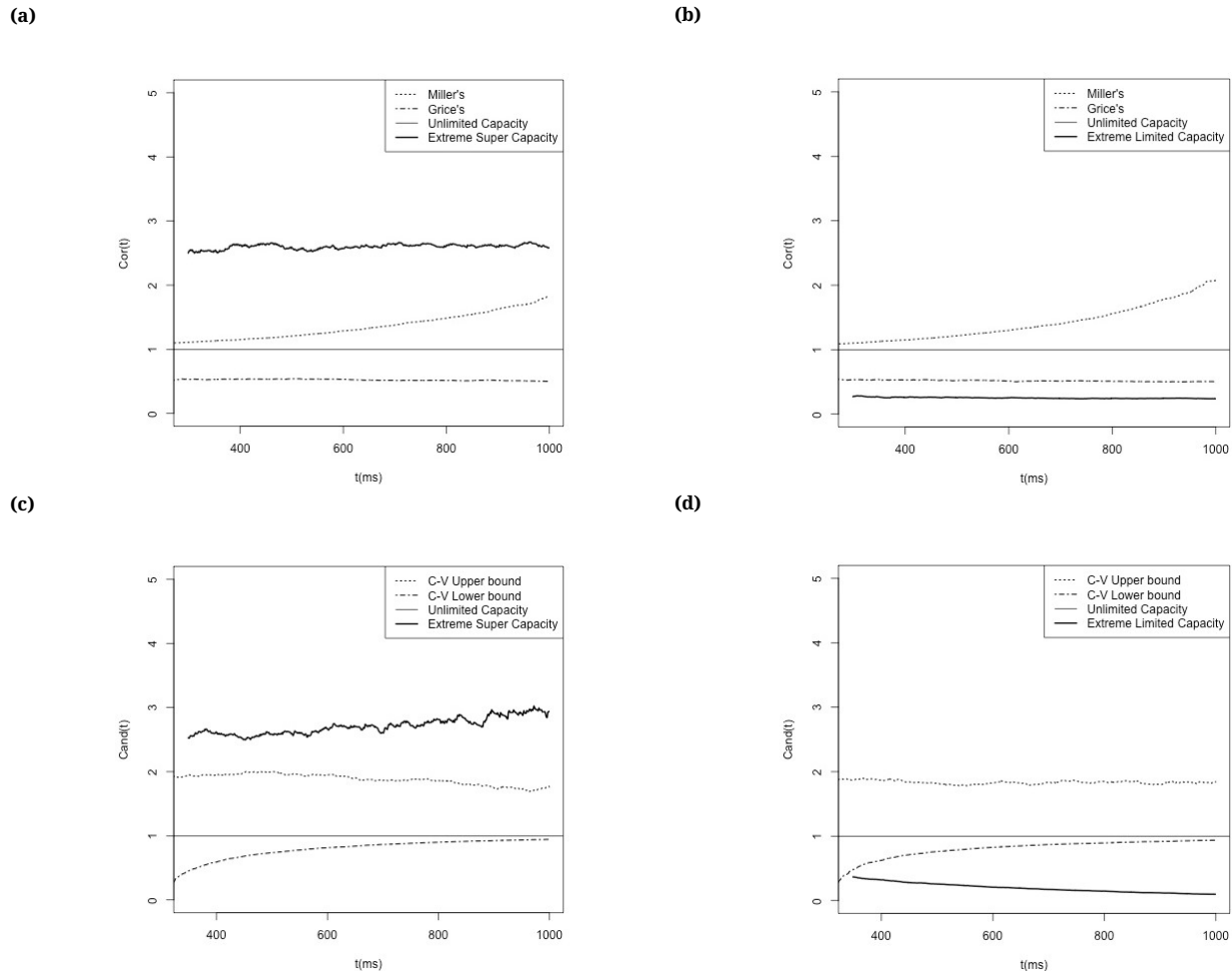
sifiable. However, we must emphasize that the stochastic linear systems approach is extremely useful for theoretical explorations and interpretations of broad sets of experimental paradigms.

So consider the OR case with varying marginals. Figure 4a presents a schematic for one of the many dynamic systems in which the marginals are not invariant. When α and β both have positive signs, it indicates mutual channel facilitation. If α and β are both negative, it indicates inhibitory connections across channels. It is, of course, possible to have, say, $\alpha > 0$ but $\beta < 0$ or vice versa, but we do not consider those arrangements here. If α and β are both 0, we retrieve the standard independent parallel self-terminating model.

Figure 4b displays the $C_{OR}(t)$ functions for two cases:

one in which α and β are 0, and one in which α and β are > 0. Here it can be seen that when the channels do not interact, performance obeys the Miller and Grice bounds, and capacity is unlimited, that is $C_{OR}(t) = 1$ for all $t > 0$. However,when the channels do interact, and the marginals do vary, performance violates the Miller bound and the capacity coefficient indicates super-capacity, that is $C_{OR}(t) > 1$. In this case, the increase in the marginals far outweighs the negative influence of the positive dependence. Figure 4c plots the contrast between non-interactive and negatively interacting channels. Here it can be seen that the negative interactions lead to performance that violates the Grice bounds and for which the capacity coefficient indicates limited-capacity processing.

Now consider the AND stopping rule with varying

Figure 4 ■ (a) A dynamic system in which the marginal distributions on processing times can vary. U$_A$ and U_B are external inputs. N_A and N_B are channels' specific noise. D is the decisional logic gate that can follow either OR or AND rule. α and β are parameters that control the amount of interactive inputs from the other channel. (b) Capacity coefficients following the OR decisional rule simulated from independent and facilitatory systems. C_{OR} of the independent channels is within the Miller and the Grice bounds, and is very close to unlimited capacity. C_{OR} of facilitatory channels violates the Miller bound at early processing time,which indicates extreme super capacity. (c) Capacity coefficients following the OR decisional rule simulated from independent and inhibitory systems. C_{OR} with inhibitory channels violates the Grice bound, which indicates extremely limited capacity in inhibitory systems.

(b)

(a)

(c)
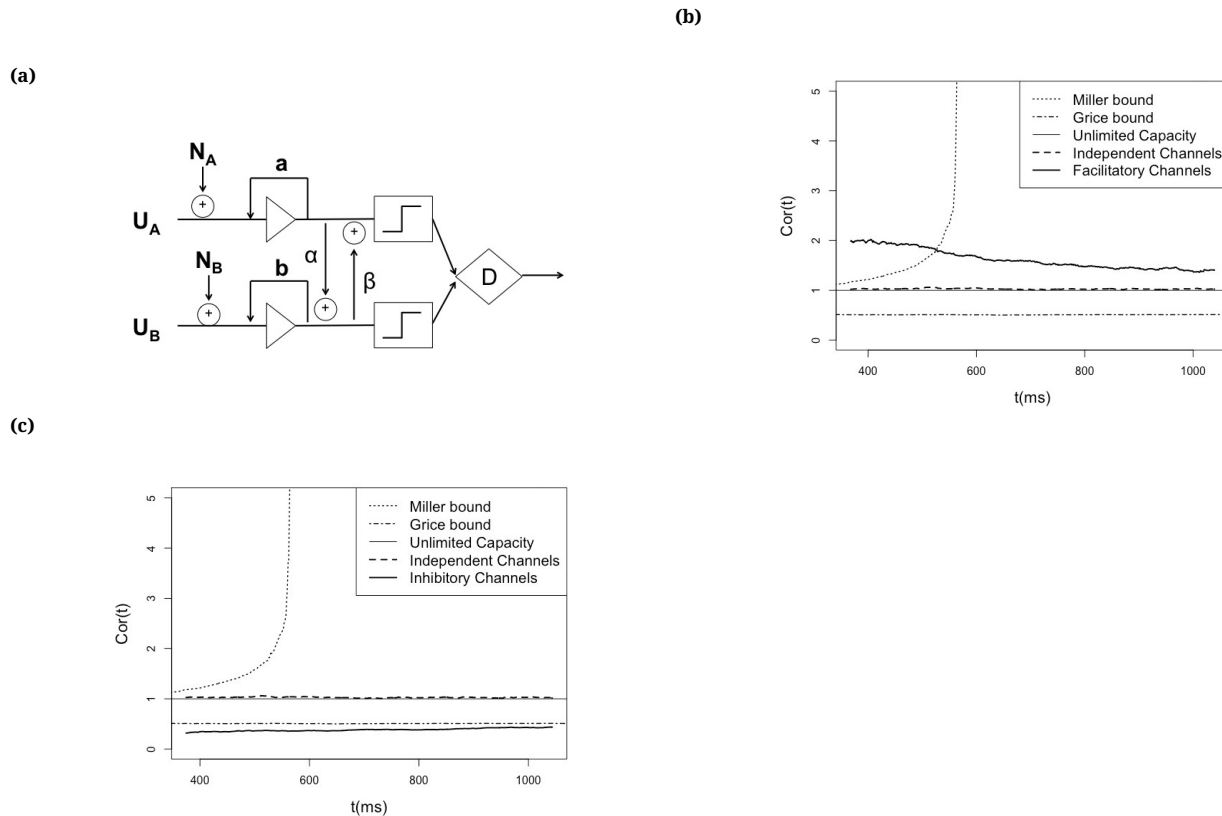
marginals with the addition of positive channel interactions. The results here are consistent with those for the OR case. In the case of positive interactions (Figure 5a), it can be seen that performance violates the Colonius-Vorberg bound and that the capacity coefficient indicates super-capacity processing. In the case of negative interactions (Figure 5b), performance falls well below the Colonius-Vorberg bound and the capacity coefficient indicates limited-capacity processing.

OR and AND Parallel Systems Permitting Variable Marginals but Obeying the Bounds

The previous section demonstrated that, as in our earlier study (Townsend & Wenger, 2004), the present class of systems can readily evoke marginal distributions that produce violation of the bounds and in the opposite direc-

tions of the original Colonius predictions. Our new simulations indicate that the general class of stochastic linear systems is quite flexible with respect to invariance or non-invariance of the marginals. In fact, it is possible for the marginals to be altered, yet obtain performances that not only obey the bounds but even lie in the province of the Colonius (1990) predictions. Figure 6a and b illustrate systems with these characteristics.

In terms of our earlier constructions, these dynamics are associated through the difference $P_{AB}(T_A \leq t) + P_{AB}(T_B \leq t) - P_{AB}(T_A \leq t$ and $T_B \leq t)$ in Equation 1 not being as large, either positive or negative, as in the preceding section. Thus, there exist accrual halting systems, based on stochastic linear systems, that can behave in a fashion consonant with the end-result predictions of Colonius and Vorberg (1994) even though the marginals
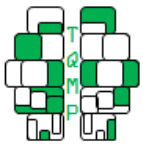
Figure 5 ■ (a) Capacity coefficients following the AND decisional rule simulated from independent and facilitatory systems with marginal variability. C_{AND} of the independent channels is within the C-V upper and lower bounds and, again, is very close to the unlimited capacity. C_{AND} with facilitatory channels violating the C-V upper bound, indicating extreme super capacity. (b) Capacity coefficients following the AND decisional rule simulated from independent and inhibitory systems. C_{AND} with inhibitory channels violating the C-V lower bound, which indicates extreme limited capacity.

(a)

(b)
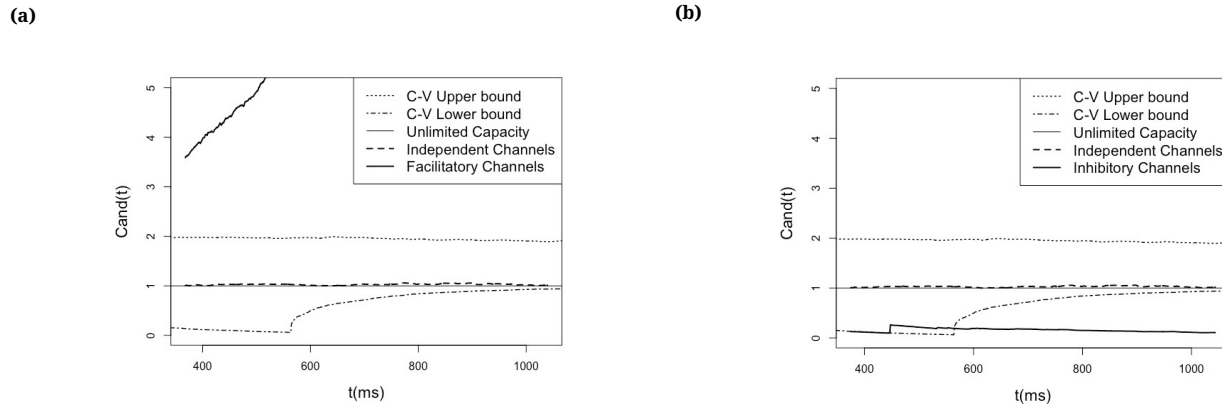
are not absolutely invariant. Further, it is straightforward to find parameters which produce moderate super capacity with positive dependence, or limited capacity with negative dependence, but which do not violate the upper or lower AND bounds. This result is elicited simply by moderating the degree of positive or negative dependence found in $P_{AB}(T_A \leq t \text{ and } T_B \leq t)$. Figure 7a and b illustrate this behavior.

Parallel Systems with Absolutely Invariant Marginals

As mentioned earlier, we were initially surprised to learn that our earlier perspective regarding dependent parallel channels (Townsend & Wenger, 2004) was based on an incorrect precept: that distributions which possess invariant marginals lie outside the class of accrual halting models. In point of fact, we have discovered a simple parallel architecture that does result in absolutely invariant marginals and for which there are no parameter settings that can elicit the Townsend and Wenger (2004) predictions.

Figure 8a thus shows a linear system with an OR decisional rule which always leaves $P_{AB}(T_A \leq t) = P_A(T_A \leq t)$ and $P_{AB}(T_B \leq t) = P_B(T_B \leq t)$ completely unchanged in moving from the single to the double stimulus condition. Figure 8b indicates that under positive channel interactions the system is limited capacity, as foretold by the Colonius interpretation, and that in the extreme case, when the dependence is extreme, the Grice bound is approached. Similarly, Figure 8c exhibits the predictions with a negative correlation due to inhibitory connections. Here processing is stochastically speeded up, as in the Colonius schema. Capacity is moderately super capacity. Although the marginals do not affect the AND predictions,

we include the AND behavior for symmetry of discussion. First consider the positively dependent interaction with the AND decisional rule: here we obtain super capacity but do not violate the Colonius upper bound (Figure 9a). Similarly, with negative dependence (Figure 9b), we obtain limited capacity but without violations of the Colonius lower bound. In sum, we suspect the elementary noisy linear model here is possibly the simplest system which leaves the marginals absolutely invariant. Such systems provide an existence proof that reasonable state space models can deliver perfect context invariance.

Moving Toward Experimental Tests: Consideration of Some AND and OR Response Time Data

The results we presented above demonstrate that the general class of interactive parallel accrual halting models are capable of predicting both invariant and varying marginals. Our developments indicate that even our relatively simple linear-systems-with-noise models can attain impressive levels of complexity. These facts raise a new set of questions about model mimicry and experimental identifiability. Specifically, this set of models, as well as models based on more complex interactions, cannot be falsified by standard experiments using AND and OR designs, at least in the usual way of implementing them in the laboratory (e. g., Colonius & Diederich, 2018; Miller, 2016; Otto & Mamassian, 2017). Yet, the question of the variance or invariance of the marginals is of significant importance. Most experiments that have attempted to measure workload capacity have run only one or the other stopping rule condition, usually the OR alternative (but c. f. Fournier, Bowd, & Herbert, 2000). Otto and Mamassian (2012) provide one
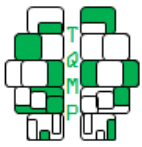
Figure 6 ■ (a) Capacity coefficients following the OR decisional rule simulated from independent and facilitatory systems with marginal variability. C_{OR} of the facilitatory system is within the Miller and the Grice bounds, but lower than the unlimited capacity coefficient. This indicates that there is moderately limited capacity in this facilitatory system. (b) Capacity coefficients following the OR decisional rule simulated from independent and inhibitory systems. C_{OR} of the inhibitory channels is within the Miller and the Grice bounds, but above the unlimited capacity coefficient.This indicates that there is moderate super capacity in this inhibitory system.

(a)

(b)
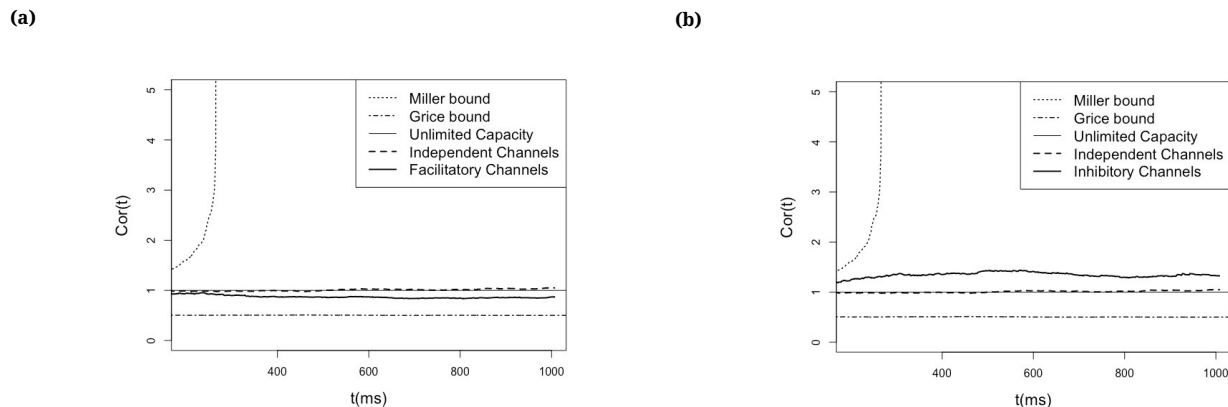
of the few studies that not only gathered data from both OR as well as AND trials. In addition they used the fits from one to make predictions about the other. Perhaps those data might be disinterred in order to examine the predicted linkages found here.

Now, consider the kind of results reported for the Townsend and Wenger (2004) simulations where the marginals were typically far from invariant. If the marginals are dominant (overpowering any increment in the joint probability) as in those explorations, then performance in both the OR and the AND conditions with positive channel dependence will likely elicit super capacity. Contrarily, with a negative dependence, with high marginal variability and therefore a major diminishment of the marginals, that situation will subdue the effect of the declined joint probability and cause limited capacity in the OR condition as well as in the AND condition. As shown above,the Colonius and colleagues' (e.g., Colonius, 1990; Colonius & Vorberg, 1994) interpretation can neither make predictions that violate the bounds nor ones where OR and AND go in the same direction with regard to limited vs.super capacity. This is a positive feature since it implies falsifiability.

On the other hand, suppose that the OR and AND experimental conditions generate qualitatively distinct capacity findings, that is OR with limited (but not extremely limited) and AND with super (but not exceedingly super) or vice versa. Then, either the Colonius-Vorberg or the modified Townsend-Wenger approximations of that behavior (Figures 6,7) could be underlying performance as indexed by the capacity coefficients. In such an instance, the ex-

perimenter might be able to render the interactions even stronger in hopes of violating the appropriate bounds, supporting the earlier Townsend and Wenger (2004) interpretation or if not, thereby buttressing the specific Colonius-Vorberg account.

Eidels, Townsend, Hughes, and Perry (2015) have contributed one of the relatively rare data sets containing both AND as well as OR capacity coefficients. Moreover, we have re-analyzed the data utilizing upgraded statistical tools (Houpt, MacEachern, Peruggia, Townsend, & Van Zandt, 2016) which were not available when the earlier study was carried out. Our present conclusions with reference to the current issues are based on these new analyses.

The OR findings were extremely consistent: All nine observers revealed limited capacity, that is $C_{OR}(t) < 1$ consistently and in a statistically significant fashion. But, $C_{OR}(t)$ never violated the Grice bound for any of the observers, though some observer's $C(t)$ functions appear to approach that bound. This result means that capacity was moderately but not extremely limited. Due to the high degree of conformity in the OR data, we plot the average capacity measures in Figure 10a. The figure includes the averaged upper OR Miller bound as well as the averaged lower Grice bound. These OR data, by themselves are not contradictory to a hypothesis of context invariance and suggest, under that interpretation, a positive correlation. Alternatively, context invariance may fail in the presence of a negative correlation but with relatively mild effects on the marginal terms.

The AND results were vastly different, as illustrated in Figures 10b and c. These figures also contain the averaged

Figure 7 ∎ (a) Capacity coefficients following the AND decisional rule simulated from independent and facilitatory systems with marginal variability. C_{AND} of the facilitatory system is within the C-V upper and lower bounds but above unlimited capacity. This indicates that this facilitatory system possesses moderate super capacity. (b) Capacity coefficients following the AND decisional rule simulated from independent and inhibitory systems. C_{AND} of the inhibitory system is within the C-V upper and lower bounds but lower than the unlimited capacity. This indicates that this inhibitory system possesses moderate limited capacity.
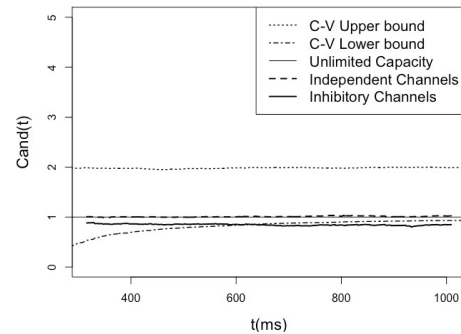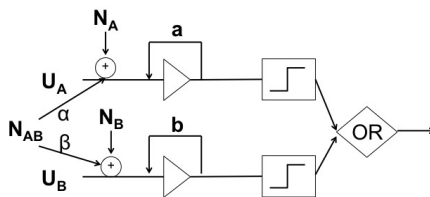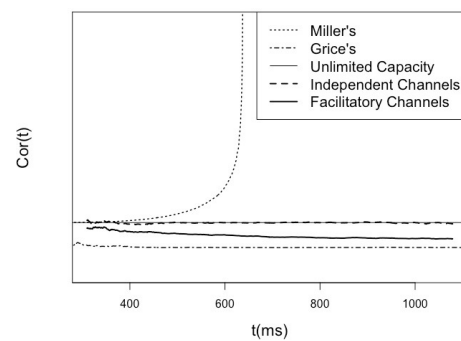
(a)
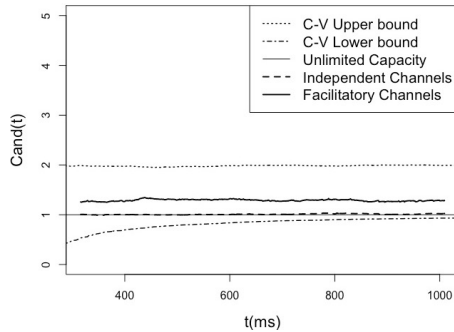
(b)

Figure 8 ∎ (a) A dynamic system obeys strict marginal invariance and follows the OR decisional rule. (b) Capacity coefficients following the OR decisional rule simulated from independent and facilitatory Colonius' systems. C_{OR} of the facilitatory channels is within the Miller and the Grice bounds, but lower than the unlimited capacity coefficient. This indicates that this facilitatory system is moderate limited capacity. (c) Capacity coefficients following the OR decisional rule simulated from independent and inhibitory Colonius' systems. C_{OR} of the inhibitory channels is within the Miller and the Grice bounds, but is above the unlimited capacity coefficient, which indicates moderate super capacity.
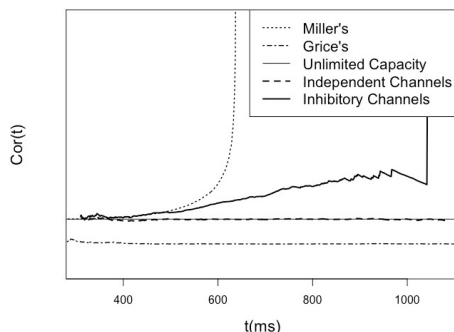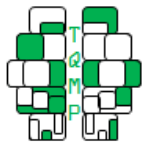
(b)

(a)

(c)

Figure 9 ■ (a) Capacity coefficients following the AND decisional rule simulated from independent and facilitatory Colonius' systems with marginal invariance. C_{AND} is within the C-V lower bound but above the unlimited capacity, which indicates moderate super capacity. (b) Capacity coefficients following the AND decisional rule simulated from independent and inhibitory Colonius' systems. C_{AND} is within the C-V lower bound but lower than the unlimited capacity, which indicates moderate limited capacity.

(a)
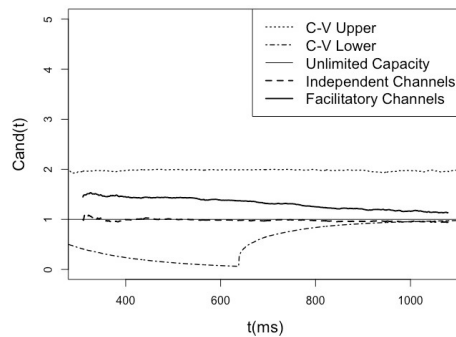
(b)
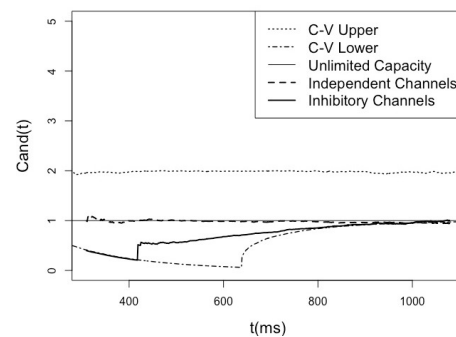
upper and averaged lower Colonius-Vorberg bounds. Four of the nine exhibited extreme limited capacity throughout their $C_{AND}(t)$ trajectories, with $C_{AND}(t)$ consistently violating the lower Colonius-Vorberg bound. In contrast, five of the nine observers revealed a transition from early moderate limitations in capacity to extreme super capacity in later portions of their data. Neither set is in line with context invariance when taken in conjunction with the OR data.

Nevertheless, we can explicate the two subsets of data in an approximative fashion with our stochastic linear systems models. Observe that we are not attempting to demonstrate the best possible fits here; rather to see if our model class can produce the very basic qualitative patterns. Thus, if a system possesses inhibitory interactions with $\alpha = \beta = -0.25$, then $C_{OR}(t)$ is found to be of moderately limited capacity but $C_{AND}(t)$ breaks the lower Colonius (1990) bound. Figure 11a and b exhibit the predictions of a model possessing negative interactions. It captures the essential patterns for both the OR as well as the AND data.

The second subset of observers evidence OR capacity which is also moderately limited but their AND capacity starts in a zone of moderately limited capacity then ascends to super capacity status. This dual pattern requires a more dramatic initiative: moderately limited OR capacity suggests either moderate across-channel inhibition if the marginals are affected more than the joint term or moderate across-channel facilitation if the marginals are (as in Colonius, 1990) less than the joint term. The initial moderately low level of AND capacity rules out the second alternative but the subsequent super AND capacity excludes the former. The next simulations explore, for the first time, a

dynamic variation in the cross-talk parameters. In this situation, α is a function of time and moves from a negative value of -0.1 up to a positive value of 0.8. Figures 11c and d exhibit behavior which roughly approximates the qualitative behavior of the designated observers.

Again, we emphasize that these developments support the view of our stochastic linear systems not as in and of themselves, specific falsifiable models. Rather they should be regarded as constituting a very general class of models which can provide a rich environment for exploration. Thus, they may prove quite helpful in both exposing key data characteristics but also in bringing forth potential explanatory dynamic models, such as those varying the input signal, that might lead to further experimental testability.

Conclusions

We have developed a comprehensive meta-theory for parallel multi-channel processing. This class, which we previously termed parallel accrual halting models (Townsend et al., 2012), encompasses a very broad set of parallel entities since it allows positive or negative interactions, even a dynamic variation of these over time. Obviously, accrual halting models lie within the broad spectrum of sequential sampling models. Our initiative here emphasizes and extends our earlier purview (e. g., Townsend & Nozawa, 1995; Colonius & Townsend, 1997; Townsend & Wenger, 2004) which uncompromisingly argues against the common practice of inferring coactivation from a violation of the race inequality. Instead, it favors viewing all interactive parallel models (including those characterized by failure of context invariance) in OR situations, as true race models.

Figure 10 ∎ (a) Estimated average capacity from the OR condition (Eidels et al., 2015).The capacity coefficient lies between the Miller bound and the Grice bound. It suggests that subjects on average, possess moderately limited capacity in the OR condition. (b) Estimated average capacity of five observers from AND condition (Eidels et al., 2015). The estimated capacity coefficient shifts from limited capacity to super capacity with increase in response time. (c) Estimated average capacity of four observers from AND condition who showed extremely limited capacities with increase in response time.

(a)

(b)

(c)
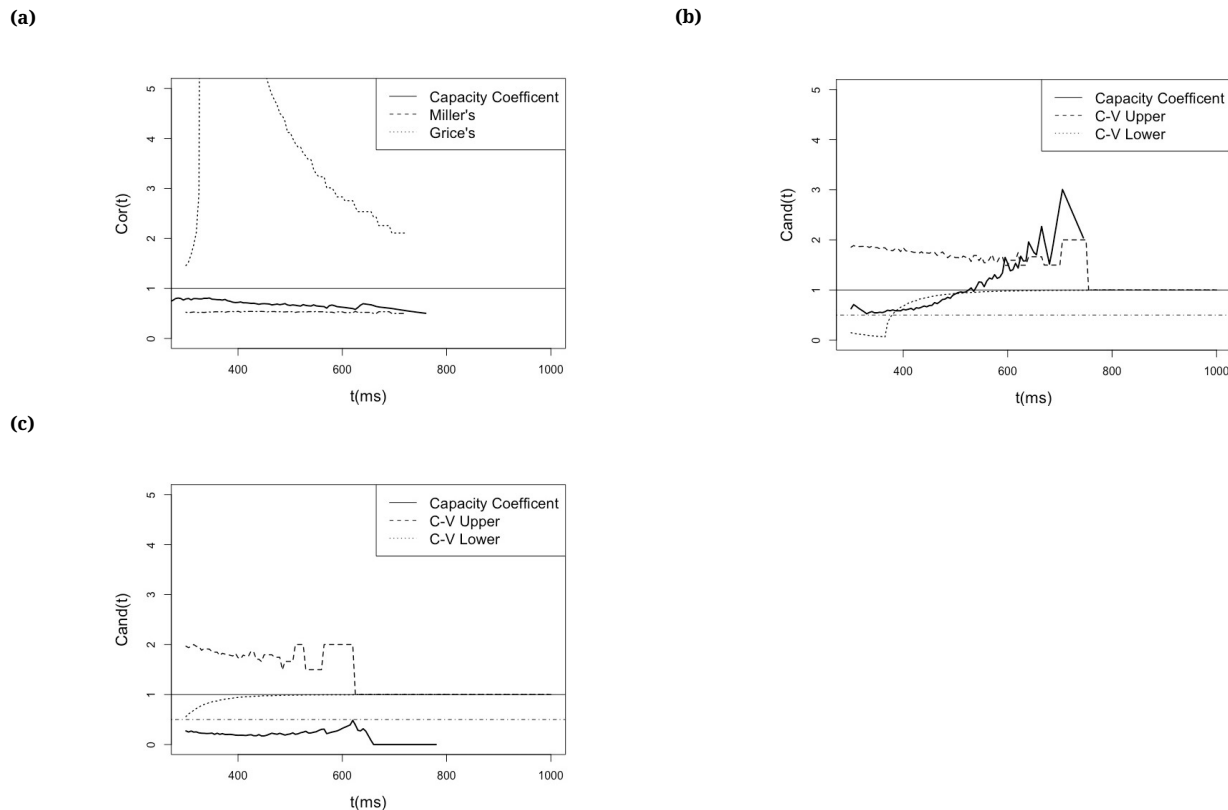
We proceeded by developing first the class of models characterized only by their joint processing times, then moving to parallel accrual halting models (i.e., with parallel sequential sampling channels, based on activation state spaces accompanied by decision criteria). Subsequently, we presented the class of parallel stochastic linear systems with decision boundaries and employed them to realize the most critical of models which did, or did not, obey context invarance and in turn, their relationship to the major OR and AND inequalities. We stated our view that this class of models is not falsifiable, at least within the usual types of paradigms.

In the course of our theoretical sojourn, we demonstrated that the crucial assumption of context invariance can be falsified if AND and OR conditions yield the same conclusions about capacity, either both super or both limited. If AND results are limited capacity and OR results are super capacity or vice versa, then a system obeying context invariance could be responsible. This logic does not require model fitting or even the assumption of sequential sampling.

However, we advocate this class as an extremely useful toolbox. First, it is highly general in that it includes all single decision boundary diffusion processes per channel (see e.g., Smith & Van Zandt, 2000). Second, being founded on stochastic differential equations, the constituent models can be prospectively employed to represent not only critical aspects of input functions but also vital characteristics of the processing filters and therefore the basic components of the processing mechanisms. Third, all of our channel models, though couched in stochastic fashion, are linear and therefore can bypass the still formidable hazards of non-linear dynamics (potential chaos being but one of these). Fourth, and quite critically, if a particular model constructed using this approach accounts well for a set of data, that model can be used to specify informative behavioral or physiological tests.

We regard the field of modeling multi-object (dimen-

Figure 11 ■ (a) Estimated moderately limited $C_{OR}(t)$ from an inhibitory interactive system with $\alpha = \beta = -0.3$. (b)Estimated extremely limited $C_{AND}(t)$ from the same inhibitory system as a. (c) Estimated moderately limited $C_{OR}(t)$ from a dynamic variation in correlation where $\alpha = \beta = -0.1$ at beginning and changes to 0.8 after simulating for 550 ms. (d) Estimated $C_{AND}(t)$ from the same system as in c, which now exhibits a limited-capacity pattern in early time and changes to super capacity later.

(a)

(b)

(c)

(d)
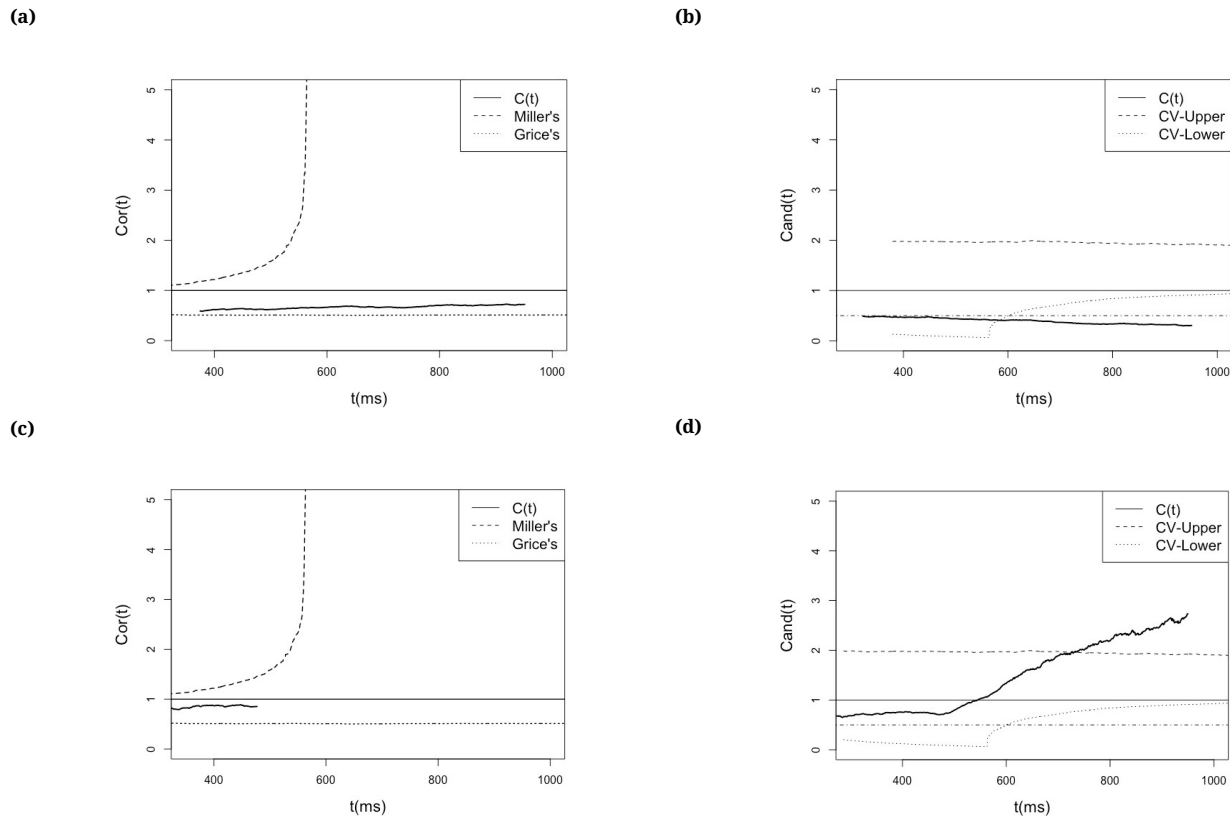
sion, feature, etc.) processing as in an embryonic phase. We envision expansion of theory, methodology, and experimentation to include other architectures, such as serial or more complex (e. g., Schweickert, 1978; Schweickert, Fisher, & Sung, 2012), continuous rather than discrete flow (Eriksen & Spencer, 1969; Townsend & Schweickert, 1989; Schweickert, 1989; Townsend & Fikes, 1995), and patterns of response frequency rather than RTs alone (e. g., Townsend & Altieri, 2012; Townsend et al., 2012).

Authors' note

Matlab code for the simuluations is available on request from JTT.

References

Algom, D., Eidels, A., Hawkins, R. X., Jefferson, B., & Townsend, J. T. (2015). Features of response times: Identification of cognitive mechanisms through mathematical modeling. *Oxford: Oxford Library of Psy-chology. The Oxford Handbook of Computational and Mathematical Psychology*, 63–98.

Algom, D., & Fitousi, D. (2016). Half a century of research on Garner interference and the separability–integrality distinction. *Psychological Bulletin*, *142*(12), 1352.

Algom, D., Fitousi, D., & Eidels, A. (2017). Bridge-building: Sft interrogation of major cognitive phenomena. In *Systems factorial technology* (pp. 115–136). Elsevier.

Ashby, F. G., & Townsend, J. T. (1986). Varieties of perceptual independence. *Psychological Review*, *93*(2), 154.

Bernstein, I. H., Rose, R., & Ashe, V. M. (1970). Energy integration in intersensory facilitation. *Journal of Experimental Psychology*, *86*(2), 196.

Besle, J., Fort, A., Delpuech, C., & Giard, M.-H. (2004). Bimodal speech: Early suppressive visual effects in human auditory cortex. *European Journal of Neuroscience*, *20*(8), 2225–2234.
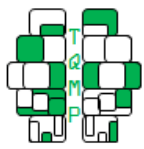
Blaha, L. M., & Houpt, J. W. (2016). Combining the capacity coefficient with statistical learning to explore individual differences. In *Mathematical models of perception and cognition, II, a festschrift for James T. Townsend, scientific psychology series* (Chap. 11). New York: Routledge.

Brown, S. D., & Heathcote, A. (2008). The simplest complete model of choice response time: Linear ballistic accumulation. *Cognitive Psychology*, *57*(3), 153–178.

Bundesen, C. (1990). A theory of visual attention. *Psychological Review*, *97*(4), 523.

Busemeyer, J. R., & Townsend, J. T. (1993). Decision field theory: A dynamic-cognitive approach to decision making in an uncertain environment. *Psychological Review*, *100*(3), 432.

Calvert, G., Spence, C., Stein, B. E., et al. (2004). *The handbook of multisensory processes*. Boston: MIT press.

Colonius, H. (1990). Possibly dependent probability summation of reaction time. *Journal of Mathematical Psychology*, *34*(3), 253–275.

Colonius, H. (2016). An invitation to coupling and copulas: With applications to multisensory modeling. *Journal of Mathematical Psychology*, *74*, 2–10.

Colonius, H., & Diederich, A. (2018). Formal models and quantitative measures of multisensory integration: A selective overview. *European Journal of Neuroscience*.

Colonius, H., & Townsend, J. T. (1997). Activation-state representation of models for the redundant-signals-effect. In *Choice, Decision and Measurement, volume in honor of R. Duncan Luce*. Mahwah, NJ: Lawrence Erlbaum Associates Publishers.

Colonius, H., & Vorberg, D. (1994). Distribution inequalities for parallel models with unlimited capacity. *Journal of Mathematical Psychology*, *38*(1), 35–58.

Diederich, A., & Colonius, H. (1991). A further test of the superposition model for the redundant-signals effect in bimodal detection. *Perception & Psychophysics*, *50*(1), 83–86.

Donkin, C., Brown, S., & Heathcote, A. (2011). Drawing conclusions from choice response time models: A tutorial using the linear ballistic accumulator. *Journal of Mathematical Psychology*, *55*(2), 140–151.

Dzhafarov, E. N. (1993). Grice-representability of response time distribution families. *Psychometrika*, *58*(2), 281–314.

Egeth, H. E. (1966). Parallel versus serial processes in multidimensional stimulus discrimination. *Perception & Psychophysics*, *1*(4), 245–252.

Eidels, A., Houpt, J. W., Altieri, N., Pei, L., & Townsend, J. T. (2011). Nice guys finish fast and bad guys finish last: Facilitatory vs. inhibitory interaction in parallel systems. *Journal of Mathematical Psychology*, *55*(2), 176–190.

Eidels, A., Townsend, J. T., & Algom, D. (2010). Comparing perception of stroop stimuli in focused versus divided attention paradigms: Evidence for dramatic processing differences. *Cognition*, *114*(2), 129–150.

Eidels, A., Townsend, J. T., Hughes, H. C., & Perry, L. A. (2015). Evaluating perceptual integration: Uniting response-time-and accuracy-based methodologies. *Attention, Perception, & Psychophysics*, *77*(2), 659–680.

Eriksen, C. W., & Spencer, T. (1969). Rate of information processing in visual perception: Some results and methodological considerations. *Journal of Experimental Psychology*, *79*(2p2), 1.

Estes, W., & Taylor, H. (1964). A detection method and probabilistic models for assessing information processing from brief visual displays. *Proceedings of the National Academy of Sciences*, *52*(2), 446–454.

Fan, J., McCandliss, B. D., Sommer, T., Raz, A., & Posner, M. I. (2002). Testing the efficiency and independence of attentional networks. *Journal of Cognitive Neuroscience*, *14*(3), 340–347.

Fitousi, D., & Algom, D. (2018). A system factorial technology analysis of the size congruity effect: Implications for numerical cognition and stochastic modeling. *Journal of Mathematical Psychology*, *84*, 57–73.

Fournier, L. R., Bowd, C., & Herbert, R. J. (2000). Interference from multi- dimensional objects during feature and conjunction discriminations. *The Quarterly Journal of Experimental Psychology Section A*, *53*(1), 191–209.

Fréchet, M. (1951). Sur les tableaux de corrélation dont les marges sont données. *Ann. Univ. Lyon, série, Sciences, Sect. A*, *14*, 53–77.

Grice, G. R., Canham, L., & Boroughs, J. M. (1984). Combination rule for redundant information in reaction time tasks with divided attention. *Perception & Psychophysics*, *35*(5), 451–463.

Gumbel, E. (1958). Extreme value statistics. New York: Columbia Univ. Press.

Harrar, V., Harris, L. R., & Spence, C. (2017). Multisensory integration is independent of perceived simultaneity. *Experimental Brain Research*, *235*(3), 763–775.

Henson, R. N., & Gagnepain, P. (2010). Predictive, interactive multiple memory systems. *Hippocampus*, *20*(11), 1315–1326.

Höffding, W. (1940). Masstabinvariante korrelationstheorie. *Schriften des Mathematischen Instituts und Instituts fur Angewandte Mathematik der Universität Berlin*, *5*, 181–233.

Houpt, J. W., MacEachern, S. N., Peruggia, M., Townsend, J. T., & Van Zandt, T. (2016). Semiparametric Bayesian approaches to systems factorial technology. *Journal of Mathematical Psychology*, *75*, 68–85.

Leone, L. M., & McCourt, M. E. (2015). Dissociation of perception and action in audiovisual multisensory integration. *European Journal of Neuroscience*, *42*(11), 2915–2922.

Link, S., & Heath, R. (1975). A sequential theory of psychological discrimination. *Psychometrika*, *40*(1), 77–105.

Logan, G. D. (1988). Toward an instance theory of automatization. *Psychological Review*, *95*(4), 492.

Lombardi, L., D'Alessandro, M., & Colonius, H. (2019). A new nonparametric test for the race model inequality. *Behavior Research Methods*, *51*(5), 2290–2301.

Luce, R. D. (1986). *Response times: Their role in inferring elementary mental organization*. Oxford: Oxford University Press.

Miller, J. (1978). Multidimensional same– different judgments: Evidence against independent comparisons of dimensions. *Journal of Experimental Psychology: Human Perception and Performance*, *4*(3), 411.

Miller, J. (1982). Divided attention: Evidence for coactivation with redundant signals. *Cognitive Psychology*, *14*(2), 247–279.

Miller, J. (1991). Channel interaction and the redundant-targets effect in bimodal divided attention. *Journal of Experimental Psychology: Human Perception and Performance*, *17*(1), 160.

Miller, J. (2016). Statistical facilitation and the redundant signals effect: What are race and coactivation models? *Attention, Perception, & Psychophysics*, *78*(2), 516–519.

Nickerson, R. S. (1973). Intersensory facilitation of reaction time: Energy summation or preparation enhancement? *Psychological Review*, *80*(6), 489.

Noel, J.-P., Modi, K., Wallace, M. T., & Van der Stoep, N. (2018). Audiovisual integration in depth: Multisensory binding and gain as a function of distance. *Experimental Brain Research*, *236*(7), 1939–1951.

Otto, T. U., & Mamassian, P. (2012). Noise and correlations in parallel perceptual decision making. *Current Biology*, *22*(15), 1391–1396.

Otto, T. U., & Mamassian, P. (2017). Multisensory decisions: The test of a race model, its logic, and power. *Multisensory Research*, *30*(1), 1–24.

Patterson, R. E., Fournier, L. R., Williams, L., Amann, R., Tripp, L. M., & Pierce, B. J. (2013). System dynamics modeling of sensory-driven decision priming. *Journal of Cognitive Engineering and Decision Making*, *7*(1), 3–25.

Raab, D. H. (1961). Forward and backward masking between acoustic clicks. *The Journal of the Acoustical Society of America*, *33*(2), 137–139.

Raab, D. H. (1962). Division of psychology: Statistical facilitation of simple reaction times. *Transactions of the New York Academy of Sciences*, *24*(5 Series II), 574–590.

Raij, T., Uutela, K., & Hari, R. (2000). Audiovisual integration of letters in the human brain. *Neuron*, *28*(2), 617–625.

Ratcliff, R. (1978). A theory of memory retrieval. *Psychological Review*, *85*(2), 59.

Schröger, E., & Widmann, A. (1998). Speeded responses to audiovisual signal changes result from bimodal integration. *Psychophysiology*, *35*(6), 755–759.

Schwarz, W., & Ischebeck, A. (1994). Coactivation and statistical facilitation in the detection of lines. *Perception*, *23*(2), 157–168.

Schweickert, R. (1978). A critical path generalization of the additive factor method: Analysis of a Stroop task. *Journal of Mathematical Psychology*, *18*, 105–139.

Schweickert, R. (1989). Separable effects of factors on activation functions in discrete and continuous models: D' and evoked potentials. *Psychological Bulletin*, *106*, 318–328.

Schweickert, R., Fisher, D. L., & Sung, K. (2012). *Discovering cognitive architecture by selectively influencing mental processes*. Singapore: World Scientific.

Shalev, L., & Algom, D. (2000). Stroop and Garner effects in and out of Posner's beam: Reconciling two conceptions of selective attention. *Journal of Experimental Psychology: Human Perception and Performance*, *26*(3), 997.

Smith, P. L., & Van Zandt, T. (2000). Time-dependent Poisson counter models of response latency in simple judgment. *British Journal of Mathematical and Statistical Psychology*, *53*(2), 293–315.

Sperling, G. (1960). The information available in brief visual presentations. *Psychological Monographs: General and Applied*, *74*(11), 1.

Sternberg, S. (1966). High-speed scanning in human memory. *Science*, *153*(3736), 652–654.

Townsend, J. T., & Fikes, T. (1995). A beginning quantitative taxonomy of cognitive activation systems and application to continuous flow processes. *Indiana University: Cognitive Science Research Report*, *131*.

Townsend, J. T. (1972). Some results concerning the identifiability of parallel and serial processes. *British Journal of Mathematical and Statistical Psychology*, *25*(2), 168–199.

Townsend, J. T., & Schweickert, R. (1989). Toward the trichotomy method: Laying the foundation of stochas-

tic mental networks. *Journal of Mathematical Psychology*, *33*, 309–327.

Townsend, J. T., & Altieri, N. (2012). An accuracy–response time capacity assessment function that measures performance against standard parallel predictions. *Psychological Review*, *119*(3), 500.

Townsend, J. T., & Ashby, F. G. (1983). *Stochastic modeling of elementary psychological processes*. Campbridge: Cambridge University Press.

Townsend, J. T., & Colonius, H. (2005). Variability of the max and min statistic: A theory of the quantile spread as a function of sample size. *Psychometrika*, *70*(4), 759–772.

Townsend, J. T., & Eidels, A. (2011). Workload capacity spaces: A unified methodology for response time measures of efficiency as workload is varied. *Psychonomic Bulletin & Review*, *18*(4), 659–681.

Townsend, J. T., Houpt, J. W., & Silbert, N. H. (2012). General recognition theory extended to include response times: Predictions for a class of parallel systems. *Journal of Mathematical Psychology*, *56*(6), 476–494.

Townsend, J. T., & Nozawa, G. (1995). Spatio-temporal properties of elementary perception: An investigation of parallel, serial, and coactive theories. *Journal of Mathematical Psychology*, *39*(4), 321–359.

Townsend, J. T., & Nozawa, G. (1997). Serial exhaustive models can violate the race model inequality: Implications for architecture and capacity. *Psychological Review*, *104*(3), 595.

Townsend, J. T., & Wenger, M. J. (2004). A theory of interactive parallel processing: New capacity measures and predictions for a response time inequality series. *Psychological Review*, *111*(4), 1003.

Townsend, J. T., & Wenger, M. J. (2014). On the dynamic perceptual characteristics of gestalten: Theory-based methods. *Oxford Handbook of Perceptual Organization*.

Townsend, J. T., Wenger, M. J., & Houpt, J. W. (2018). Uncovering mental architecture and related mechanisms in elementary human perception, cognition, and action. *Stevens' Handbook of Experimental Psychology and Cognitive Neuroscience*, *5*, 1–30.

Wenger, M. J., Schuster, C., & Townsend, J. T. (2002). *Testing capacity hypotheses: Applying proportional hazards and frailty models*. Paper presented at the *35th Annual Meeting of the Society for Mathematical Psychology*, Miami University of Ohio, Oxford, Ohio.

Wenger, M. J., & Townsend, J. T. (2006). On the costs and benefits of faces and words: Process characteristics of feature search in highly meaningful stimuli. *Journal of Experimental Psychology: Human Perception and Performance*, *32*(3), 755.

Zhang, R., Liu, Y., & Townsend, J. T. (2018). A theoretical study of process dependence for standard two-process serial models and standard two-process parallel models. In *Invariances in Human Information Processing* (pp. 117–142). Routledge.

Zhang, R., Liu, Y., & Townsend, J. T. (2019). A theoretical study of process dependence for critical statistics in standard serial models and standard parallel models. *Journal of Mathematical Psychology Special Issue on Systems Factorial Technology*.

Appendix: Simulation Methods

The model used for simulations follows a stochastic dynamic linear system approach used in Townsend and Wenger (2004). The systems use this basic form:
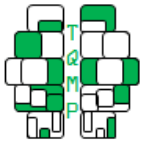
$$\frac{d}{dt}\mathbf{X}(t) = \mathbf{A}\mathbf{X}(t) + \mathbf{B}\mathbf{U}(t) + \mathbf{H}(t)$$
$$\mathbf{Y}(t) = \mathbf{C}\mathbf{X}(t) + \mathbf{D}\mathbf{U}(t)$$

Here, $\mathbf{A}$ is the activation matrix and determines how information within each channel is accumulated. In our simulations, we fixed

$$A = \left[\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right],$$

so that the dynamic system is stable and no interaction between channels is introduced through channel accumulations. $\mathbf{B}$ is the channel activation coefficient matrix and determines how much the input information goes into each channel. In our simulations, we fixed $\mathbf{B} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$ so that inputs selectively influence correspondent channels. $\mathbf{U}$ is the input matrix containing input values to each channel. $\mathbf{H}$ is the matrix of noise power determining how much noise goes into each channel, $\mathbf{H} = \left[\begin{array}{c} \eta_1 \\ \eta_2 \end{array} \right]$. The noise power η_i can be transformed into variance by the following:

$$\frac{\eta_i}{t_c} = (\sigma\sqrt{\Delta T})^2$$

where t_c is the sampling rate of the simulation, which was set to 0.001 in our simulations. $\mathbf{C}$ is the channel distributed coefficient matrix determining how much accumulated information from each channel goes to outputs. $\mathbf{D}$ is the input activated coefficient matrix for corresponding channels' outputs. In our simulations, $\mathbf{D}$ is fixed at $\mathbf{D} = \left[\begin{array}{c} 0 \\ 0 \end{array} \right]$. In addition, there is a criterion γ_i for each channel. Criteria were set to be the same across channels in our current simulations.

Interactions between channels were introduced through either $\mathbf{H}$ matrix for cases where context invariance holds, or the $\mathbf{C}$ matrix for cases in which it does not. Specifically, in cases where context invariance holds, the correlations between channels were introduced into the system by a common source. In this case, we set

$$\mathbf{C} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]$$

and $\eta_1 \sim N(0, (\sigma_1^2 + \alpha^2 \sigma_c^2)\Delta T)$ and $\eta_2 \sim N(0, (\sigma_2 + \beta^2 \sigma_c)\Delta T)$ where σ_1 and σ_2 are the standard deviations of each of the channel noise, and σ_c is the standard deviation of the common noise. If the interaction is positive, we set $\alpha = \beta = 0.2$. If the interaction is negative, we set $\alpha = \beta = -0.4$. In cases where context invariance was violated, we set $\eta_1 \sim N(0, \sigma_1 \Delta T)$ and $\eta_2 \sim N(0, \sigma_2 \Delta T)$ so that there was no longer any interaction between channels introduced through common noise. The cross-channel interactions were introduced into the system by manipulating the off-diagonal elements of $\mathbf{C}$. For positive interactions, we set the off-diagonal elements to be 0.5, and for negative interactions, we set them to be equal to -0.5.

Citation

Townsend, J. T., Liu, Y., Zhang, R., & Wenger, M. J. (2020). Interactive parallel models: No Virginia, violation of Miller's race inequality does not imply coactivation and yes Virginia, context invariance is testable. *The Quantitative Methods for Psychology, 16*(2), 192–212. doi:10.20982/tqmp.16.2.p192