2021 ■ Vol. 17 ■ no. 2

The exact binomial test between two independent proportions: A companion

Louis Laurencelle [a]

[a] Université d'Ottawa

Abstract ■ This note, a summary in English of a former article in French (Laurencelle, 2017), presents an outline of the theory and implementation of an exact test of the difference between two binomial variables. Essential formulas are provided, as well as two executable computer programs, one in Delphi, the other in R.

Keywords ■ exact test of two proportions. **Tools** ■ Delphi, R.

Acting Editor ■ Denis Cousineau (Université d'Ottawa)

Reviewers ■ No reviewer

louis.laurencelle@gmail.com

10.20982/tqmp.17.2.p076

Introduction

Let there be two samples of given sizes n_1 and n_2, and and numbers x_1 and x_2 of "successes" in corresponding samples, where $0 \leq x_1 \leq n_1$ and $0 \leq x_2 \leq n_2$. The purpose of the test is to determine whether the difference $x_1/n_1 - x_2/n_2$ results from ordinary random variation or if it is probabilistically exceptional, given a probability threshold. Treatises and textbooks on applied statistics report a number of test procedures using some form of normal-based approximation, such as a Chi-squared test, various sorts of z (normal) and t (Student's) tests with or without continuity correction, etc. (Howell, 2017; Siegel & Castellan, 1988; Sokal & Rohlf, 1995). The reader may also be aware of Fisher's "exact probability test", which is indeed an exact probability test but not a test applicable to independent proportions, the rows (n_1, n_2) and columns (x_1, x_2) totals being both fixed. Liddell (1978) proposes a test with an exact probability calculation, but an incomplete one that uses a single point estimate of the parametric probability for the null hypothesis of the test.

The test procedure of Liddell (1978) exploits a point estimate of π, the probabilistic reference parameter, which is the maximum likelihood estimator for two binomial series, i.e., here, $\hat{\pi} = (x_1 + x_2)/(n_1 + n_2)$. Using this single estimate and following a suggestion of Kendall and Stuart (1977), Liddell lists all possible values of the two proportions and their differences, comparing these with the observed difference and then accumulating the probabilities of equal or more extreme differences.

Our proposed test, similar to that of Liddell (1978), uses the same procedure of enumerating the $(n_1+1) \times (n_2+1)$ possible differences, this time exploiting the integral domain $(0..1)$ of the π parameter in combination with an appropriate weighting function.

Derivation of the exact test

Let $R_1 = (x_1, n_1)$ and $R_2 = (x_2, n_2)$ be the results of two binomial observations, and their observed difference $d_O = x_1/n_1 - x_2/n_2$, where $R_1 \sim B(x_1|n_1, \pi_1)$, $R_2 \sim B(x_2|n_2, \pi_2)$, with π_1 and π_2 being the two unknown binomial probability parameters. The null hypothesis of no parametric difference stipulates that $\pi_1 = \pi_2 = \pi_0$. Under the null hypothesis and applying a one-sided test, what is the probability that $d \geq d_O$?

The distribution of the random variable $d = y_1/n_1 - y_2/n_2$ ($0 \leq y_1 \leq n_1$, $0 \leq y_2 \leq n_2$) is statistically conditioned on two factors: the size parameters n_1 and n_2, which are given, and the two observed proportions x_1/n_1 and x_2/n_2, with respect to which the likelihood of the underlying π_0 value will vary.

With any given value of π_0, the probability sought for is $P\{d \geq d_O\} = p(H_0)$ is :

$$p(\pi_0) = \sum_{y_1, y_2} b(y_1|n_1, \pi_0) \times b(y_2|n_2, \pi_0), \quad (1)$$

where $b(y|n,\pi)$ is the binomial probability given by $\binom{n}{y}\pi^y(1-\pi)^{n-y}$. The sum is computed using every value of y_1 and y_2, $0 \le y_1 \le n_1$, $0 \le y_2 \le n_2$, for which

$$y_1/n_1 - y_2/n_2 \ge x_1/n_1 - x_2/n_2 \qquad (2)$$

(for a right-side difference test). However. as pointed out above, any value of π_0 is not as likely as another, considering the observed, real values of binomial variables x_1 and x_2. Indeed, the likelihood of a given π_0 value with respect to the observed data is simply

$$L(\pi_0|x_1,n_1;x_2,n_2) = {\pi_0}^{x_1+x_2}{\pi_0}^{n_1+n_2-x_1-x_2}. \qquad (3)$$

This function, akin to the Beta function (Johnson, Kotz, & Balakrishnan, 1995), has here its maximum value at $\hat{\pi} = (x_1+x_2)/(n_1+n_2)$, the so-called maximum likelihood estimate used in Liddell (1978) test procedure.

Our procedure, which encompasses the whole domain of π_0, consists in weighting the summation (1) by the likelihood function (3), resulting in the following formula (see Laurencelle, 2017, for more details):

$$P\{d_{random} \ge d_O\} = \frac{(N+1)!n_1!n_2!}{(2N+1)!X!\,(N-X)!}\sum_{y_1,y_2}{}^{*}\frac{(X+y_1+y_2)!\,(2N-X-y_1-y_2)!}{y_1!\,(n_1-y_1)!y_2!\,(n_2-y_2)!} \qquad (4)$$

where $N = n_1 + n_2$, $X = x_1 + x_2$, and the right-hand summation $\sum^*$ of (4) is restricted to (y_1, y_2) pairs such that $y_1/n_1 - y_2/n_2 \ge x_1/n_1 - x_2/n_2$ for a right-side difference test, and vice-versa for a left-side test.

In Listing 1 and Listing 2 are provided two computer codes, one for `Pascal/Delphi` language, the other for `R` language, which perform the computation of $p\{d_{random} \ge d_O\}$. For a two-sided test, state the probability $\min(2 \times p, 1)$, p begin likely to overflow $1/2$ due to the discrete nature of the distribution of d.

Two examples

Below are two data sets illustrating the results of formula (4) for the difference between two independent proportions, together with the results of some other test procedures. The classical Chi-squared and z-tests (without continuity correction) need not be presented. As for the Anscombe (1948), Chanter (1975) and Freeman and Tukey (1950) tests, here F-T, they are normal z tests using Fisher's arcsine normalizing transformation of proportions, each offering an original variant. A full-scale study of these and other variants is coming soon ("The statistical treatment of proportions including analysis of variance, with examples", in French in this journal).

Authors' note

The author wishes to thank the Editor for his helpful and clarifying suggestions.

References

Anscombe, F. J. (1948). The transformation of poisson, binomial and negative-binomial data. *Biometrika*, *35*, 246–254. doi:10.1093/biomet/35.3-4.246

Chanter, D. O. (1975). Modifications of the angular transformation. *Journal of the Royal Statistical Society Series B (Applied Statistics)*, *24*, 354–359. doi:10.2307/2347101

Freeman, M. F., & Tukey, J. W. (1950). Transformations related to the angular and the square root. *Annals of Mathematical Statistics*, *21*, 607–611. doi:10.1214/aoms/1177729756

Howell, D. C. (2017). *Fundamental statistics for the behavioral sciences (9th ed.)* Washington: Cengage Learning.

Johnson, N. L., Kotz, S., & Balakrishnan, N. (1995). *Continuous univariate distributions (vol. 2) (2nd ed)*. New York: Wiley.

Kendall, M. G., & Stuart, A. (1977). *The advanced theory of statistics, vol. 1*. New York.

Laurencelle, L. (2017). Le test binomial exact de la différence entre deux proportions et ses approximations / the exact binomial test between two proportions and its approximations. *The Quantitative Methods for Psychology*, *13*(2), 141–147. doi:10.20982/tqmp.13.2.p141

Liddell, D. (1978). Practical tests of 2?2 contingency tables. *The Statistician*, *25*, 295–304. doi:10.2307/2988087

Siegel, S., & Castellan, N. J. (1988). *Nonparametric statistics for the behavioral sciences*. New York: McGraw-Hill.

Sokal, R. R., & Rohlf, F. J. (1995). *Biometry (3rd ed.)* New York: Freeman.

Table 1 ■ Two examples of computations

x_1/n_1	x_2/n_2	p_1	p_2	d_O	p_{Exact}	p_{χ^2}, p_Z	p_{Anscombe}	p_{Chanter}	p_{F-T}
13/20	15/29	0.650	0.517	0.133	0.183	0.178	0.183	0.183	0.186
13/112	89/473	0.188*	0.116*	0.072	0.032	0.035	0.031	0.030	0.033

Note. * The programs being coded for right-tail extreme probability evaluation, the (x_1/n_1) and (x_2/n_2) data are automatically swapped so that $p_1 \geq p_2$.

Appendix: The programs

Listing 1: Original Pascal / Delphi program

```
program difference_between_two_independent_proportions;
  {$APPTYPE CONSOLE}
  Uses SysUtils,Math;
  Const nmax=10000; { >= 2 * (n1 + n2) }
  var x1, x2, n1, n2, N, X, R, i, y1, y2: integer;
     lf: array[0..10000] of real;
     p1, p2, dif, lcons, lprob, sum: real;
begin
  writeln('L. Laurencelle, July 2005, 2017');
  writeln('Exact right-tail probability of the difference between two proportions');
  {to avoid overflow, precompute log of the factorials }
  lf[0]:=0;
  for i:=1 to nmax do lf[i]:=lf[i-1]+ln(i);
  repeat writeln;
    write('x1 n1 : ');readln(x1,n1);
    write('x2 n2 : ');readln(x2,n2);
    if x1/n1 < x2/n2 then begin y1:=x1; x1:=x2; x2:=y1; y1:=n1; n1:=n2; n2:=y1 end;
    p1:=x1/n1; p2:=x2/n2; dif:=p1-p2;
    write('p1 =',p1:6:3,'  p2 =',p2:6:3,'  dif =',dif:6:3);
    if abs(p1-p2)<=0 then writeln('  pExact = 0.5') else
      begin
        N:=n1+n2; X:=x1+x2; R:=N-X;
        {Selective summation by enumeration}
        sum:=0;
        lcons:=lf[N+1]+lf[n1]+lf[n2]-lf[2*N+1]-lf[X]-lf[N-X];
        for y1:=0 to n1 do for y2:=0 to n2 do
          if y1/n1-y2/n2>=dif then
            begin
            lprob:=lf[X+y1+y2]+lf[2*N-X-y1-y2]-lf[y1]-lf[n1-y1]-lf[y2]- lf[n2-y2];
            sum:=sum+exp(lcons+lprob)
            end;
        write('  pEXACT =',sum:8:4);
      end
  until false
end.
```

Listing 2: R implementation

```
# pExact computes the exact probability of a difference larger than
# the observed difference between the two proportions x1 over n1 vs. x2 over n2.
# L. Laurencelle, July 2005, 2017
pExact <- function(x1, n1, x2, n2) {
  if ((x1>n1)|(x2>n2)) stop("Invalid input: x1 (x2) must be smaller than n1 (n2")

  # all computations are in log to avoid overflow
  lf = lfactorial # log(factorial(x)) = ln(x!)

  if ((x1/n1)<(x2/n2)) {# swap the values
    temp <- x1; x1 <- x2; x2 <- temp
    temp <- n1; n1 <- n2; n2 <- temp
  }

  # if equal proportions, then p is 0.5
  if (x1/n1 == x2/n2) {
    return(0.5)
  } else {
    N = n1+n2
    X = x1+x2
    total = 0
    lcons = lf(N+1)+lf(n1)+lf(n2) -lf(2*N+1)-lf(X)-lf(N-X)
    for (y1 in 0:n1) for (y2 in 0:n2)
      if (round( (y1/n1 - y2/n2) - (x1/n1 - x2/n2), 10)>=0) {
        lprob = lf(X+y1+y2)+lf(2*N-X-y1-y2)-lf(y1)-lf(n1-y1)-lf(y2)-lf(n2-y2)
        total = total + exp(lcons + lprob)
      }
    return(total)
  }
}
```

Citation

Laurencelle, L. (2021). The exact binomial test between two independent proportions: A companion. *The Quantitative Methods for Psychology*, *17*(2), 76–79. doi:10.20982/tqmp.17.2.p076

Received: 07/05/2021 ∼ Accepted: 14/05/2021